MTS 101 LECTURE 4 : MATRICES AND MATRIX ALGEBRA

DEFINITION

A matrix is a rectangular array of the elements of a field (i.e. an array of numbers). Thus if m, n are two positive integers ≥ 1 and F is a field (ℝ or ℂ) then the array:

\[
\begin{pmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix}
\]

is called an \( m \times n \) matrix in F (since it contains m rows and n columns).

Its first row is \((a_{11} \ a_{12} \ldots a_{1n})\) and first column is \( a_{11} \ a_{12} \ldots \)

The numbers that constitute the matrix are called its ELEMENTS.

Let \( a_{ij} \) denote the element of the matrix in the \( i^{th} \) row and \( j^{th} \) column. Then for ease of notation we can denote our \( m \times n \) matrix by

\[
(a_{ij}) \quad i = 1,2,\ldots,n \quad j=1,2,\ldots,m \text{ or simply by capital letter } A_{m \times n}
\]

Order

The order of a matrix is the number of rows and columns e.g \((a_{ij})\) is of order \( m \times n \).

If \( m = n \), then the matrix is called a SQUARE MATRIX of order \( n \).

Definition 2: Row and Column Matrices
A rectangle matrix consisting of only a single row is called a ROW MATRIX e.g. (1,2,3,4). Similarly, a rectangle matrix consisting of a single column is called a COLUMN MATRIX e.g.

\[
\begin{pmatrix}
3 \\
5 \\
7 \\
4
\end{pmatrix}
\]

**Definition 3: Null Matrix**

This is a matrix having each of its elements = 0 e.g. 

\[
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

**Definition 4: Diagonal Element, Diagonal Matrix**

The elements \(a_{ij}\) of a matrix \((a_{ij})\) are called its DIAGONAL ELEMENTS (or elements of the main diagonal)

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\]

\(a_{11} = 1, a_{22} = 5, a_{33} = 9\)

A square matrix in which all the elements other than the diagonal elements are zero is called a DIAGONAL MATRIX.

\[
\begin{pmatrix}
d_1 & 0 & 0 & 0 & \ldots & 0 \\
0 & d_2 & 0 & 0 & \ldots & 0 \\
0 & 0 & d_3 & 0 & \ldots & 0 \\
- & - & \ldots & - & \ldots & 0 \\
0 & 0 & 0 & 0 & \ldots & d_n
\end{pmatrix}
\]

Viz \(=\)

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 \\
0 & 0 & 9 & 0 \\
0 & 0 & 0 & 6
\end{pmatrix}
\]

Such a matrix is denoted \((d_1, d_2, \ldots, d_n)\) or \((d_i, d_{ik})\) for \(i, k = 1, 2, \ldots n\) where \(d_{ii}=1, d_{ik} = 0 (i \neq k)\)
NB: Its diagonal elements may also be zero

\[
\begin{align*}
\text{i. } & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\text{ii. } & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\text{iii. } & \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}
\end{align*}
\]

are diagonal matrices.

**Definition 5: Scalar and Scalar Matrix**

A diagonal matrix where diagonal elements are all equal is called a SCALAR MATRIX.

\[
\begin{align*}
\text{e.g. } & \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \\
& \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
& \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{bmatrix}
\end{align*}
\]

**Definition 6: Identity Matrix (or Unit Matrix)**

A diagonal matrix whose elements are each equal to 1 is called and IDENTITY MATRIX. It is denoted \(I_{nxn}\).

\[
\begin{align*}
\text{e.g. } & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{align*}
\]

**Definition 7: Symmetry**

A square matrix where elements are arranged symmetrically about the main diagonal is called a SYMMETRIC MATRIX.

\[
\begin{align*}
\text{e.g. } & \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}
\end{align*}
\]

On the other hand, if for a square matrix, there is no symmetric about the main diagonal but for every element \(a_{ij}\) on one side of the main diagonal, there is a corresponding \(-a_{ij}\) on the other side, then the matrix is a SKEW-SYMMETRIC MATRIX.

Furthermore, the diagonal elements are all zero
\[
\begin{pmatrix}
0 & 1 & 2 & 3 \\
-1 & 0 & -\frac{1}{2} & 2 \\
-2 & \frac{1}{2} & 0 & 1 \\
-3 & 2 & -1 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
0 & h & g \\
-h & 0 & f \\
-g & -f & 0
\end{pmatrix}
\]

Definition 8: Triangular matrix

A square matrix whose elements \(a_{ij}\) are all zero whenever \(i < j\) is called a LOWER TRIANGULAR MATRIX.

A square matrix whose elements \(a_{ij} = 0\) whenever \(i > j\) is called an UPPER TRIANGULAR MATRIX.

Hence, a diagonal matrix is both upper and lower matrix.

e.g. \[
\begin{pmatrix}
1 & 0 \\
2 & 0 \\
0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 \\
2 & 4 & 0 \\
3 & 3 & 2
\end{pmatrix}, \quad \text{→ Lower Triangular Matrix}
\]

\[
\begin{pmatrix}
3 & 4 & 2 \\
0 & 1 & \frac{1}{2} \\
0 & 0 & \frac{3}{4}
\end{pmatrix}, \quad \begin{pmatrix}
0 & \frac{1}{2} \\
0 & 3
\end{pmatrix}, \quad \text{→ Upper Triangular Matrix}
\]

MATRIX ALGEBRA

Equality of Matrices

A and B are equal if

i. they are of the same order

ii. their corresponding elements are the same
Addition of Matrices

If A and B are of the same order, the their sum is a matrix C of the same order whose elements are the sums of the corresponding elements of A and B.

\[ C = A + B \]

\[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \]

\[ C_{ik} = a_{ik} + b_{ik} \quad i = 1,2, \ldots \ldots m, \quad k = 1,2,\ldots \ldots n \]

* Only matrices of the same order can be added.

Properties of Matrix Addition

i. Matrix addition is commutative \[ A + B = B + A \]

ii. Matrix addition is Associative \[ (A + B) + C = A + (B + C) \]

iii. If 0 is a null matrix of the same order as A, the \[ A + 0 = 0 + A = A \]

iv. To each A there exists a matrix B of the same order s.t \[ A + B = 0 = B + A \]

\[ (i) \rightarrow (iv) \Rightarrow \text{Matrix addition is Abelian} \]

Exercise 1.

Find the sum of these matrices and establish their Commutativity

i. \[ \begin{bmatrix} 1 & 2 \\ -3 & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & -3 \\ 7 & 5 \end{bmatrix} \]

ii. \[ \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \]

Exercise 2:

Establish the associativity of the following matrices

i. \[ \begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 1 \\ 3 & 5 & 7 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 & 4 \\ 3 & 0 & 6 \\ 2 & 5 & 4 \end{bmatrix} \]
Exercise 3:

Establish the order of each matrix 1, 2 and 4 and find the $a_{11}$, $a_{12}$, etc what are the diagonal elements.

Exercise 4:

Which of the following matrices are (i) Triangular Matrices, (ii) Unit Matrices (iii) null matrices and Scalar matrices.

```
i. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} 
ii. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} 
iii. \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 6 & 2 & 0 \end{bmatrix} 
iv. \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} 

v. \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} 
vi. \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} 

vii. \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} 
viii. \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 5 \end{bmatrix} 

ix. \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} 
``` 

Exercise 5:

i. If $A = \begin{bmatrix} 1 \\ 4 \\ 2 \\ 5 \\ 3 \\ 6 \end{bmatrix}$ Find a Matrix $B$ such that $A + B = 0$

ii. If $A = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 4 \\ 6 \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ -3 \\ -5 \\ -4 \\ -8 \end{bmatrix}$ Find $A + B$

**Multiplication by Scalar**

Let $A = (a_{ij})$ i = 1, 2, 3, ... m be a matrix, j = 1, 2, 3, ... n

And Let $\alpha$ be a scalar (i.e. any number), then $\alpha A = C = (C_{ij})$

Where $\alpha a_{ij} = C_{ij}$ i = 1, 2, ....m, j = 1, 2, ....n

**Example**
If \[
\begin{bmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
\end{bmatrix}
= \begin{bmatrix}
4a_1 & 4a_2 & 4a_3 \\
4b_1 & 4b_2 & 4b_3 \\
\end{bmatrix}
\]

Properties

If A, B are matrices and α, β are scalars, then

i. \( α(A + β) = αA + αβ \)

ii. \( (α + β)A = αA + βB \)

iii. \( (αβ) A = α(βA) \)  

Examples to be given in class

DIFFERENCE OF TWO MATRICES

If two matrices A and B are of the same order, then the difference \( A - B = A + (-B) = A + (-1)B \)

Exercise 6:

i. For Exercise (2) above, find \( (A - B) + C \)

ii. \( A - B - C \)  

iii. \( 2A + 3B \)  

iv. \( A + 2B + \frac{1}{2}C \)

Exercise 7:

If \( A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \\ 3 & 1 & 2 \end{bmatrix} \) and \( B = \begin{bmatrix} \end{bmatrix} \)

i. Find a matrix C such that \( A + C \) is a diagonal matrix.

ii. Find a matrix D such that \( A + B = 2D \).

iii. Find a Matrix E such that \( (A + B) + E \) is zero matrix.

MULTIPLICATION OF MATRICES

The product \( AB \) of two matrices exist if the number of columns of A = the number of rows of B.
Since $A_{2\times 2}$ and $B_{2\times 3}$ i.e, no. of column of $A =$ no. of rows of $B$, then $AB$ exist.

Let $A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}$ of order $m \times n$ and $B = \begin{bmatrix} b_{11} & \ldots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{n1} & \ldots & b_{nq} \end{bmatrix}$ of order $n \times q$

Then $AB$ is the matrix

$$C = \begin{bmatrix} C_{1i} & C_{1q} \\ C_{mi} & C_{mq} \end{bmatrix}$$ of order $m \times q$

in which the element $C_{ij}$ is the sum of products (term by term) of elements of $i^{th}$ row of $A$ and the $j^{th}$ column of $B$. Thus for the matrices $A = (a_{ik})$, $B = (b_{kj})$, the product $AB$ is matrix $C = (C_{ij})$

where $C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

* Multiplication is possible if no. of column of the first matrix = no. of rows of the second matrix.

Example:

If $A = \begin{bmatrix} 2 & 5 & 3 \\ 0 & 2 & 1 \\ -1 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -5 & 0 \\ 0 & 2 & 6 \end{bmatrix}$ Find (i). $AB$, (ii) $BA$ (iii) $A^2$ (iv) $5B^2$