1 Quadratic Equations

1.1 Roots of a quadratic equation

The general form of a quadratic equation is

\[ ax^2 + bx + c = 0 \]  \hspace{1cm} (1.1)

where \( a, b, c \) are constants and \( a \neq 0 \).

The roots of the quadratic equation (1.1) is given by

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a} \]

where \( D = b^2 - 4ac \).

The nature of the roots of a quadratic equation is determined by the value of \( D \).

(i) If \( D > 0 \), the equation will have two different real roots.

(ii) If \( D = 0 \), the equation has two equal roots.

(iii) If \( D < 0 \), the equation has complex roots.

1.2 Sum and product of the roots

Let \( \alpha \) and \( \beta \) be the roots of the quadratic equation (1.1), then it is equivalent to the equation

\[ (x - \alpha)(x - \beta) = 0 \]

or

\[ x^2 - (\alpha + \beta)x + \alpha\beta = 0 \]  \hspace{1cm} (1.2)

Dividing (1.1) through by \( a \), we have

\[ x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \]  \hspace{1cm} (1.3)

Comparing (1.2) and (1.3), we obtain

\[ \alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a} \]  \hspace{1cm} (1.4)

Using (1.4), we can find the sum and the product of the roots directly from the coefficients in the quadratic equation (1.1).
Example 1. If the roots of \( x^2 - x + 1 = 0 \) are \( \alpha \) and \( \beta \), find \( \alpha + \beta \) and \( \alpha \beta \).

Comparing the given equation with (1.1), \( a = 1, b = -1, c = 1 \). Hence
\[
\alpha + \beta = 1
\]
and
\[
\alpha \beta = 1
\]

Example 2. Construct an equation with roots \( \sqrt{5} + 2, \sqrt{5} - 2 \).

Let \( \alpha = \sqrt{5} + 2 \) and \( \beta = \sqrt{5} - 2 \). Then
\[
\alpha + \beta = \sqrt{5} + 2 + \sqrt{5} - 2 = 2\sqrt{5}
\]
\[
\alpha \beta = (\sqrt{5} + 2)(\sqrt{5} - 2) = 1
\]
Using (1.2), the equation is
\[
x^2 - 2\sqrt{5}x + 1 = 0.
\]

Example 3. If \( \alpha \) and \( \beta \) are the roots of the equation \( ax^2 + bx + c = 0 \), obtain in terms of \( a, b \) and \( c \) the values of (i) \( \frac{1}{\alpha} + \frac{1}{\beta} \) (ii) \( \alpha - \beta \).

We express (i) and (ii) in terms of \( \alpha + \beta \) and \( \alpha \beta \).

(i) \[
\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha \beta} = \frac{-b}{c} = -\frac{b}{c}.
\]

(ii) \[
\alpha - \beta = \pm \sqrt{(\alpha + \beta)^2 - 4\alpha \beta} = \pm \sqrt{(-\frac{b}{c})^2 - 4\frac{c}{a}} = \pm \sqrt{\frac{b^2}{a^2} - \frac{4c}{a}}
\]
Hence, \( \alpha - \beta = \pm \frac{1}{2} \sqrt{b^2 - 4ac} \).

Example 4. If \( \alpha, \beta \) are the roots of the equation \( 3x^2 - x - 5 = 0 \), form the equation whose roots are \( 2\alpha - \frac{1}{\beta}, 2\beta - \frac{1}{\alpha} \).

From the given equation, \( a = 3, b = -1 \) and \( c = -5 \). Thus,
\[
\alpha + \beta = \frac{1}{3}, \quad \alpha \beta = -\frac{5}{3}.
\]
Given the roots $2\alpha - \frac{1}{\beta}$ and $2\beta - \frac{1}{\alpha}$,

$$2\alpha - \frac{1}{\beta} + 2\beta - \frac{1}{\alpha} = 2(\alpha + \beta) - \frac{\alpha + \beta}{\alpha\beta}$$

$$= 2\left(\frac{1}{3}\right) - \left(\frac{1}{3} \times \frac{3}{5}\right)$$

$$= \frac{2}{3} + \frac{1}{5}$$

$$= \frac{10 + 3}{15}$$

$$= \frac{13}{15}$$

$$(2\alpha - \frac{1}{\beta})(2\beta - \frac{1}{\alpha}) = 4\alpha\beta + \frac{1}{\alpha\beta} - 4$$

$$= -\frac{169}{15}$$

Therefore the required equation is

$$x^2 - \frac{13}{15}x - \frac{169}{15} = 0$$

that is

$$15x^2 - 13x - 169 = 0.$$  

**Exercise:** One root of the equation $2x^2 + bx + c = 0$ is three times the other root. Show that $3b^2 = 32c$. 

2 Cubic Equations

2.1 Introduction

The general form of a cubic equation is

\[ ax^3 + bx^2 + cx + d = 0 \] (2.1)

where \(a, b, c\) and \(d\) are constants, \(a \neq 0\).

Equation (2.1) is also expressible as

\[ x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0 \] (2.2)

If \(\alpha, \beta, \gamma\) are roots of the cubic equations (2.1), (2.2), then

\[ x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} \equiv (x - \alpha)(x - \beta)(x - \gamma) \]

\[ = x^3 - \frac{\alpha + \beta + \gamma}{\alpha \beta \gamma}x^2 + \frac{\alpha \beta + \beta \gamma + \gamma \alpha}{\alpha \beta \gamma}x - \frac{\alpha \beta \gamma}{\alpha \beta \gamma} \]

Thus comparing coefficients,

\[ \alpha + \beta + \gamma = -\frac{b}{a} \]

\[ \alpha \beta + \beta \gamma + \gamma \alpha = \frac{c}{a} \]

\[ \alpha \beta \gamma = -\frac{d}{a} \]

Thus the equation whose roots are \(\alpha, \beta, \gamma\) is

\[ x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha \beta + \beta \gamma + \gamma \alpha)x - \alpha \beta \gamma = 0 \] (2.3)

2.2 Useful identities and examples

\[ \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha \beta + \beta \gamma + \gamma \alpha) \]

\[ \alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha \beta + \beta \gamma + \gamma \alpha) + 3\alpha \beta \gamma. \]

**Example 2.1.** If \(\alpha, \beta\) and \(\gamma\) are the roots \(x^3 - 7x + 1 = 0\), find the equation whose roots are \(\alpha^2, \beta^2, \gamma^2\).
Solution: From the given equation

\[
\begin{align*}
\alpha + \beta + \gamma &= 0 \\
\alpha\beta + \beta\gamma + \gamma\alpha &= -7 \\
\alpha\beta\gamma &= 1
\end{align*}
\]

Thus,

\[
\begin{align*}
\alpha^2 + \beta^2 + \gamma^2 &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \\
&= 0 - 2(-7) \\
&= 14
\end{align*}
\]

\[
\begin{align*}
\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 &= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2(\alpha\beta\gamma + \alpha^2\beta\gamma + \alpha\beta\gamma^2) \\
&= (-7)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) \\
&= 49 - 2(1)(0) \\
&= 49
\end{align*}
\]

\[
\begin{align*}
\alpha^2\beta^2\gamma^2 &= (\alpha\beta\gamma)^2 = 1^2 = 1.
\end{align*}
\]

Hence, the required equation is

\[
x^3 - 14x^2 + 49 - 1 = 0.
\]

Exercise

1. If \(\alpha, \beta\) and \(\gamma\) are the roots of \(x^3 - 3x + 1 = 0\), find the value of \(\alpha^3 + \beta^3 + \gamma^3\).

2. If \(\alpha, \beta\) and \(\gamma\) are the roots of the equation \(ax^3 + bx^2 + cx + d = 0\), where \(a \neq 0\), show that

\[
(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 = \frac{(c^2 - 2bd)}{a^2}.
\]
3 Simultaneous Equations

3.1 Simultaneous linear equations in two variables

The general form of simultaneous linear equations in two variables is

\[ ax + by = c \]
\[ dx + ey = f \]

where \( x, y \) are variables, \( a, b, c, d, e, f \) are constants.

Various methods exist for solving these equations for \( x \) and \( y \). These are:

(i) elimination method
(ii) substitution method
(iii) matrix method and
(iv) graphical method.

The reader is encouraged to find out.

3.2 Simultaneous equations, at least one non-linear

The general form of simultaneous equations in which one is linear one is quadratic is

\[ ax + bx = c \]
\[ dx^2 + e x y + fy^2 = g \]

where \( x, y \) are variables, and \( a, b, c, d, e, f, g \) are arbitrary constants.

Example: Solve the simultaneous equations for \( x \) and \( y \).

\[ x^2 + y^2 = 25 \quad (1) \]
\[ x + 3y = 5 \quad (2) \]

From (2), \( x = 5 - 3y \).
Substitute for \( x \) in (2),
\[ (5 - 3y)^2 + y^2 = 25 \]
\[ y^2 - 3y = 0 \]
\[ y(y - 3) = 0 \]
either \( y = 0 \) or \( y = 3 \).
Hence, \( x = 5 \) when \( y = 0 \) or \( x = -4 \) when \( y = 3 \).