COURSE CODE: **MTS 105 LECTURE NOTE**
COURSE TITLE: **ALGEBRA FOR BIOLOGICAL SCIENCES**
NUMBER OF UNIT: **03**
LECTURER: **DR. I. O. ABIALA**

COURSE OUTLINE: Elementary set theory: set notations, set operations, algebra of sets, Venn diagram and Applications. Operations with real numbers: indices, Logarithms, surds, uses of Logarithms in agricultural sciences. Remainder and factor theorems. Partial fractions, Linear and Quadratic inequalities. Theory of quadratic equations, Cubic equations, equations reducible to quadratic type. Sequences and series: Arithmetic and Geometric progression, Arithmetic and Geometric Mean, Arithmetic and Geometric series, nth term of a series, Binomial theorem, Binomial series, the general term of Binomial series. Matrix algebra: Matrices, algebra of matrices, Determinant of a matrix, properties and inverse of a matrix, solution of a linear system of equations. Elementary trigonometry: Degree and Radian measures, Pythagorean identities, Trigonometric functions of any angle, graphs, inverse trigonometric functions, compound angles and solution of trigonometric equations.

REFERENCES

(1) **E.Egbe, G.A. Odili and O.O. Ugbebor:** Further Mathematics published by Africana- FEB Publishers, Gbagada, Lagos, 2000.

SET THEORY

Definition: A set is a collection of objects or things that is well defined.

Here are some examples of sets:

1. A collection of students in form one
2. Letters of the alphabet
3. The numbers 2, 3, 5, 7, and 11
4. A collection of all positive numbers
5. The content of a lady's purse

The concept of set is very important because set is now used as the official mathematical language. A good knowledge of the concept of set is, therefore, necessary if mathematics is to be meaningful to its users.

Notation

A set is usually denoted by capital letters; while the objects comprising the set are written with small letters. These objects are called members or elements of a set.

For example set A has members a, b, c, d.

Convention

The listing of a set A as $a, b, c, d,$ as seen above is not an acceptable mathematical specification of a set. The correct representation of a set that is listed is to write the elements, separated by commas and enclosed between braces or curly brackets.

e.g., set $A = \{\ a, b, c, d.\ \}$.

The statement b is an element or member of set A or b belongs to a' is written in the manner $b \in A.$ The contrary statement that b does not belong to A is written as: $b \notin A.$

There are two ways of specifying a set. One way is by listing the elements in the set, such as:

$A = \{\ a, b, c, d.\ \}$.

A second way of specifying a set is by stating the rule or property which characterizes the set.

For example, $B = \{\ x/2 < x < 5.\ \}$or $B = \{\ x/2 < x < 5.\ \}$. Notice, the stroke/or colon: can be used interchangeably, with each as 'such that'. The representation, $B = \{\ x/2 < x < 5.\ \}$ is read as follows:
B is a set consider of elements $x,$ such that 2 is less than x and x is less than 5.

If a set is specified by listing its elements, we call it the tabular form of a set; and if it is specified by stating its property, such as $C = \{\ x/x$ is odd $\}$, then it is called the set builder form.

Finite and Infinite Sets

A finite set is one whose members are countable: for example, the set of students in Form 1. Other examples are:

(i) the contents of a lady's hand-bag;
(ii) whole numbers lying between 1 and 10;
(iii) members of a football team.

The finite set is itself in exhaustive; readers can give other examples of a finite set.

An infinite set is one whose elements are uncountable, as they are infinitely numerous. Here are a few examples of the infinite sets;

(i) Real numbers.
(ii) Rational numbers
(iii) Positive even numbers
(iv) Complex numbers

The main distinction between a finite set and that a finite set has a definite beginning and a definite end, while the infinite set may have a beginning and no end or vice versa or may not have both beginning and end.

For example, we specify the set of positive even numbers, as follows:

$P = \{\ 2,4,6,...\ \}$ or
$P = \{\ x : x > 2, x \text{ is even }\ \}$

The set of real whole numbers which end with the number 3 is written as follows:

SUBSETS

Suppose $P = \{\ a, b, c, d, e, f\ \}$ and $Q = \{\ c, d, e.\ \}$, then we say Q is contained in P, and we use symbol $'\subset'$ to denote the statement 'is contained in', or 'is subset of'. Thus $Q \subset P$, is ready as $'Q$ is contained in P'. More aptly put, Q is contained in P if there is an x, such that $x \in Q$ implies $x \in Q$. The statement Q is contained in P can be put in reverse order as $'P$ contains Q' and we write $P \supset Q$. However, this form is not very popular. If Q is not a subset of set $R = \{\ 3,4,a\ \}$, then we write $Q \not\subset R$. It should be noted that unless every member or Q is also a member of P, then can we say Q is subset of P.

EQUITY OF SETS

Two sets of X and Y are equal if and only if $X \subset Y$ and $Y \subset X$. Suppose $X = \{\ 1,2,3\ \}$ and $Y = \{\ 3,1,2\ \}$ and $X = Y$. Note that the rearrangement of the elements if a set does not alter the set.

TYPES OF SETS

Null or Empty Sets

Null means void, therefore, a null set is an empty set, or a set that has no members. The null set is denoted by the symbol $\{\ \}$. Note that $\{0\}$ cannot be classified as a null set, because it has an element, zero.

Singleton

Any set which has only one member is called a singleton. e.g., $(\ a\ )$ is a singleton.

The Universal Set

Set is a subset of a larger set is called the universal set or empty, the **Universe of Discourse.**

Thus, in any given context, the total collection of elements under discussion is called the Universal set.

The symbol U or E is often used to denote a universal set. For example, if we toss a die, once, we expect to have either $1,2,3,4,5,$ or $6,$ as an end result. If there are no other expected results different from this numbers, then we say, for this particular experiment, the universal set is $\{\ 1,2,3,4,5,6\ \}$. Thus a universal set is the total population under discussion.

Proper Subsets

If P is a subset of Q and if there is at least one member of Q which is not a member of P, then P is a proper subset of Q and we write $P \subset Q$.

Consider the set $A = \{\ 1,2,3,\ \}$. The following sets $\{\ 1,2,3,\ \},\{\ 1,2,\ \},\{\ 1,,3,\ \},\{\ 2,3,\ \},\{\ 1,\ \}\{\ 2,\ \},\{\ 3,\ \},\{\quad\}$ are subsets of A. The set $\{\ 1,2,3\ \}$is not a proper subset of A; whereas all others including $\{\quad\}$are proper subsets of A. Thus $\{\ 1,2,3\ \} \not\subset \{\ 1,2,3\ \}$, but $\{\ 1,2,3\ \}$, $\{\ 1,2,3\ \} \subseteq \{\ 1,2,3\ \}$.

Power Set

The collection of all the subsets of any set S is called the power set of S. If a set has n members, where n is finite, then the total number of subsets of S is 2^n. Occasionally we denote the power set S by 2^S.

For example: Let $A = \{a, b, c\}$. The subsets of A are $\{a,b,c\}, \{a,b\}$, $\{a,c\}, \{b,c\}$, $\{a\}, \{b\}, \{c\}$ and $\{ \}$. The power set of A written $p(A) = 2^3$ subsets; as seen above.

Example: Find the power set 2^S of the sets

(a) $S = \{ 3,4)$
(b) $S = \{ a, \{ ,1,2 \} \}$

Solution:

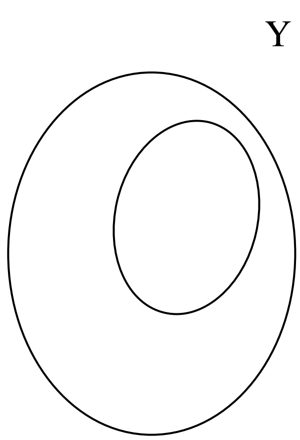
(a) $S = \{ 3,4 \}, \{ 3 \}, \{ 4 \}, \{ \},$
(b) $2^P = \{ a, \{ ,1,2 \} \}, \{ a \}, \{ 1,2 \}, \{ \}$

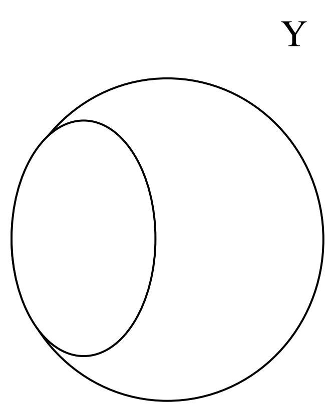
In this example, (b) contains only two elements a and $\{ 1,2 \}$

Venn – Euler Diagrams

The theory of set can be better understood if we make use of the Venn-Euler diagrams. The Venn-euler diagram is an instructive illustration which depicts relationship between sets.

Suppose $X \subset Y$ and $X \not\subset Y$, we can represent this statement in a Venn-Euler diagram as follows:






A
B

Set Operations

In set, we use the symbols $\cup$ read 'unions and $\cap$ read 'intersection' as operations. These operations are similar but not exactly the same as the operations in arithmetic. At the end of this chapter, a reader of this topic should be able to identity areas of analogy between the operation in arithmetic and those of a set.

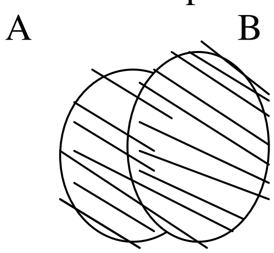
Union of Sets

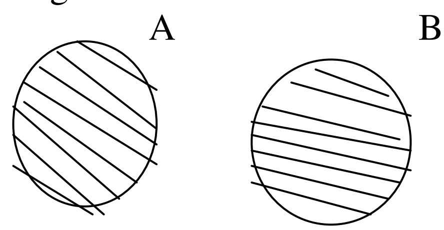
Definition

The union of sets A and B is the set of all elements which belongs to A or B or to both A and B. This is usually written as $A \cup B$, and read 'A union B'.

In set language, we define $A \cup B$ as:
$A \cup B = \{ x : x \in A \text{ or } \in B \}$.

The shaded portions in the Venn-Euler diagram in $A \cup B$

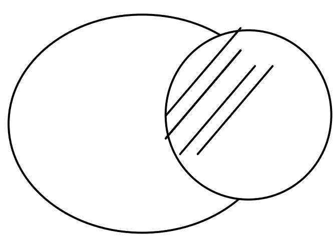




The Intersection of Sets

The intersection of sets $A and B$ is the set of elements which belong to both $A and B$. Simply, 'A intersection B' written $A \cap B$ consists of elements which are common to both $A and B$.

The Venn-Euler diagram which represent $A \cap B$ is shaded portion.



In set language

$A \cap B = \{ \ x : x \in A and x \in B$

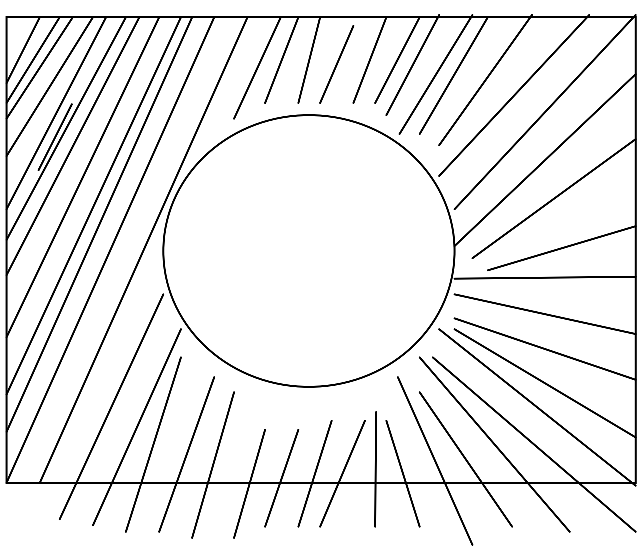
Complement of Sets

The complement of a set x is the set of elements which do not belong to x, but belong to the universal set. The complement of a set x is usually represented by x' or $x^{c.}$

The complement of x' or x^{c}.

The complement of x is represented in the Venn-Euler diagram

U



x

In set language, $A^{c} = x : x \in U, x \notin A$

The Algebra of Sets

The operations of union $\cup$ are loosely analogous to those of addition and multiplication in number algebra. By this token we can apply the laws of algebra conveniently to sets without loss of generality.

The Closure property

If X and Y are sets which are subsets of the universal set U then the following hold:

$X \cup Y \subset U$ and $X \cap Y \subset U$.

The analogy in number algebra; using those operations of $+$ and x are $2+3=5 \in R$ and $2x3=6 \in R$; where R is the real number system. If the addition or multiplication of 2 and 3 gives some number that cannot be found in the real number system R, we say the operation of $+$ or x is not closed.

Similarly in set theory, the operations of union and intersection are closed.

The Commutative Law
$X \cup Y = Y \cup U \quad X \cap Y = Y \cap X$. Parallel examples in arithmetic are $2+3=3+2$ and $2X3=3X2$.

Thus any two sets are commutative with respect to $\cup$ and $\cap$.

The Associative Law
$X \cup (Y \cup Z) = (X \cup Y)UZ$ and
$X \cap (Y \cap Z) = (X \cup Y) \cap Z$

Again, sets obey the associative law.

The Identity
In every day arithmetic, $0+1=1+0=1$ and $3X1=1X3=3$, are two correct solutions. The zero, in the first case is called the additive identity; while 1 in the second case is called the multiplicative identity.
By a similar analogy , every set has quantities $\{\ \}$ and U with the property that:

(i) $X \cup \{\ \} = \{\ \} \cup X = X$
(ii) $X \cap U = U \cap X = X$

Thus, $\{\ \}$ is the identify with respect to union $\cup$ and U is the identity with respect to intersection $\cap$.

Inverse
In the set of real number R,
$a+(-a)=(-a)+a=0$ and $aXa^1=a^1Xa=1$. This a number x operated on its inverse gives identity. i.e., x Inverse $=$identity.
Similarly in set theory, every set has an inverse with respect to the operations of $\cup$ and $\cap$

(i) $X \cup X^{'} = X'U = U$ and
$X \cup U = U \cup X = U$

(iii) $X \cap X' = X' \cap X = \{ \ \}$ and
$X \cap \{ \ \} = \{ \ \} X' X = \{ \ \}$

The distributive Law

$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ and
$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$
The operation of union is distributive over the operation of intersection and vice versa.

The Laws of complementation

(i) $X \cup X' = \cup$ and
(ii) $(X') = X$
(iii) $(X \cup Y)' = X \cap Y'$
(iv) $X \cap Y)' = X' \cup Y'$
(ii)– (iv) are called de Morgan's Laws

<u>Number set</u>
<u>Definition</u>: Let $A=\{3,4,6,8,9,10\}$ be a set, then we say set A has 6 elements, and We write $n(A)=6$. If $B=\{2,3,4,5,6\}$, then $n(B)=5$ and $n(A \cup B)=8$.
_Therefore, if A and B are disjoint,

$n(A \cup B) = n(A) + n(B)$
If A and B are not disjoint
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
This could be verified using the Venn diagram that

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B)$$
$$- n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

APPLICATION OF SET THEORY

The concept of the set theory could be applied to solve technological, industrial, economic, social, educational and political problems.

Example 1: During the orientation programme of the newly admitted students of the Federal University of Agriculture, Abeokuta, 18 students ate white-rice, 25 ate fried-rice, 23 ate beans, 9 ate white and fried-rice, 10 ate fried-rice and beans and 6 ate white-rice and beans. If there were 50 students altogether and 5 students did not take any of the three meals. How many students ate

(i) all the meals
(ii) only fried-rice
(iii) beans but not fried-rice
(iv) white-rice and beans but not fried-rice

solution (1st method) :Let W,F, and B denote the students who ate white-rice, fried-rice and beans respectively in the following Venn diagram. Let x denote those who ate all the three meals and μ, the universal set.

Venn diagram

From the Venn diagram, we have

$n(F \cap B \cap W') = 10 - x$

$n(W \cap F \cap B') = 9 - x$

$n(W \cap B \cap F') = 6 - x$

Therefore,

$n(W \cap F' \cap B') = [18 - (9 - x) - (6 - x) - x]$

$= 3 + x$

Similarly,

$n(F \cap W' \cap B') = [25 - (9 - x) - (10 - x) - x]$

$= 6 + x$

and

$n(B \cap W' \cap F') = [23 - (6 - x) - (10 - x) - x]$

$= 7 + x$

Recall that the total number of students who did not eat any of the three meals is 5, hence, the number of the students that ate one or two or the three meals =50-5=45. Therefore,

$45 = (3 + x) + (9 - x) + x + (6 - x) + (6 + x) + (10 - x) + (7 + x)$

$\Rightarrow 45 = 41 + x$

$\Rightarrow x = 4$

Hence,

(i) 4 students ate the three meals
(ii) 10 students ate only fried-rice
(iii) 13 students ate beans but not fried-rice
(iv) 2 students ate white-rice and beans but not fried-rice.

<u>2nd method</u>

Given that n(W)=18, n(F)=25, n(B)23

$n(W \cap F) = 9$

$n(F \cap B) = 10$

$n(W \cap B) = 6$

$n(W \cup F \cup B) = 50 - 5 = 45$

Using the relation

$n(W \cup F \cup B) = n(W) + n(F) + n(B) - n(W \cap F)$

$- n(W \cap B) - n(F \cap B) + n(W \cap F \cap B)$

Therefore,

$45 = 18 + 25 + 23 - 9 - 6 - 10 + n(W \cap F \cap B)$

$\Rightarrow n(W \cap F \cap B) = 4$

Which provides solution to (i), others can also be obtained using (i).

<u>Example 2:</u> A survey of the number of university students who read the three newspapers was conducted. Their preferences for the new Nigerian, the Daily Times and the Pioneer newspapers are given below:

48% read the New Nigerian
26% read the daily Times
36% read the Pioneer
8% read the New Nigerian and Pioneer
5% read the daily Times and Pioneer
4% read the New Nigerian and the Daily Times
4% read none of the newspapers

Calculate the percentage of those who read

(i) all three newspapers

(ii) only the Pioneer

(iii) only the New Nigerian

Solution: (i) Let N,D and P denote the percentage of the students who read the New Nigerian, the Daily Times and the Pioneer newspapers respectively. Suppose 100 students were interviewed, then

n(N)=48, n(D)=26, n(P)=36

$n(N \cap P) = 8$

$n(D \cap P) = 5$

$n(N \cap D) = 4$

Since 4% of the students read neither N,D, nor P, then

$$n(N \cup D \cup P) = 100 - 4 = 96$$
$$96 = 48 + 26 + 36 - 8 - 5 - 4 + n(N \cap D \cap P)$$
$$\Rightarrow 96 = 93 + n(N \cap D \cap P)$$
$$\Rightarrow n(N \cap D \cap P) = 3$$

Hence, 3 students read all the newspapers.

(iii) since $n(N \cap D \cap P) = 3$ and $n(D \cap P) = 5$ therefore

$$n(N' \cap D \cap P) = 5 - 3 = 2$$
$$n(N \cap D' \cap P) = 8 - 3 = 5$$
$$\Rightarrow n(P) = 36 - 5 - 3 - 2 = 26$$

Therefore, 26% read only Pioneer

(iii) those who read only the New Nigerian =48-5-3-1=37%

SURDS

Definition 1: Any number which can be expressed as a quotient $\frac{m}{n}$ of two integers ($n \neq 0$), is called a rational number. Any real number which is not rational is called irrational. Irrational numbers which are in the form of roots are called **surds**. For example, $\sqrt{2}, \sqrt{3}, \sqrt{5}, \pi$ and $3\sqrt{2}$ are irrational numbers while $\sqrt{16}, 3\sqrt{8}$ and $5\sqrt{32}$ can be expressed in rational form.

Definition 2: A general surd is an irrational number of the form $a\sqrt[n]{b}$, where a is a rational number and $\sqrt[n]{b}$ is an irrational number , while $\sqrt[n]{\ }$ is called a radical.

RULES FOR MANIPULATING SURDS

(i) $a\sqrt{b}+c\sqrt{b}=(a+c)\sqrt{b}$. This is the addition law of surds with the same radicals.

(ii) $a\sqrt{d}-c\sqrt{d}=(a-c)\sqrt{d}$. This is the subtraction law of surds with the same radicals.

(iii) $\sqrt{ab}=\sqrt{a}.\sqrt{b}$.

(iv) $(a\sqrt{b}).(c\sqrt{d})=ac\sqrt{bd}$.

(v) $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$

(vi) $(a\sqrt{b})\div(c\sqrt{d})=\frac{a}{c}\sqrt{\frac{b}{d}}$

(vii) $(\sqrt{a})^2=a=\sqrt{a^2}$

(viii) $(\sqrt{a})^n=\sqrt{a^n}$

(ix) $\sqrt{a^{-m}}=\frac{1}{\sqrt{a^m}}$

(x) $\frac{1}{\sqrt{a^{-m}}}=\sqrt{a^m}$

Simplification of surds

Example; Simplify the following (i) $\sqrt{75}$ (ii) $\sqrt{80}$ (iii) $\sqrt{18}$ (iv) $\sqrt{60}$

Solution: Using rule 3

(i) $\sqrt{75}=\sqrt{25\times 3}=\sqrt{25}.\sqrt{3}=5\sqrt{3}$

(ii) $\sqrt{80}=\sqrt{16\times 5}=\sqrt{16}.\sqrt{5}=4\sqrt{5}$

(iii) $\sqrt{18}=\sqrt{9\times 2}=\sqrt{9}.\sqrt{2}=3\sqrt{2}$

(iv) $\sqrt{60}=\sqrt{4\times 15}=\sqrt{4}.\sqrt{15}=2\sqrt{15}$

Addition and subtraction of surds

Example; Simplify the following (i) $\sqrt{50}-\sqrt{18}+\sqrt{32}$ (ii) $\sqrt{80}+\sqrt{20}-\sqrt{45}$ (iii) $\sqrt{28}+\sqrt{63}$

Solution: Using rule 1 and 2

(i) $\sqrt{50}-\sqrt{18}+\sqrt{32}$

$=\sqrt{25\times 2}-\sqrt{9\times 2}+\sqrt{16\times 2}$

$=5\sqrt{2}-3\sqrt{2}+4\sqrt{2}$

$=6\sqrt{2}$

(ii) $\sqrt{80}+\sqrt{20}-\sqrt{45}$
$=\sqrt{16\times5}+\sqrt{4\times5}-\sqrt{9\times5}$
$=4\sqrt{5}+2\sqrt{5}-3\sqrt{5}$
$=3\sqrt{5}$
(iii) $\sqrt{28}+\sqrt{63}$
$=\sqrt{4\times7}+\sqrt{9\times7}$
$=2\sqrt{7}+3\sqrt{7}$
$=5\sqrt{7}$

Rationalization of surds

A surd of the form $\frac{\sqrt{3}}{2}$ cannot be simplified, but $\frac{2}{\sqrt{3}}$ can be written in a more convenient form. Then, we multiply the numerator and denominator of $\frac{2}{\sqrt{3}}$ by $\sqrt{3}$. Such that $\frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$. This process is called rationalization.

Useful hints on rationalization of surds

(i) $\sqrt{a}.\sqrt{a}=a$
(ii) $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$
(iii) $(x\sqrt{a}+y\sqrt{b})(x\sqrt{a}-y\sqrt{b})=x^2a-y^2b$
(iv) $(x+y\sqrt{b})(x-y\sqrt{b})=x^2-y^2b$
(v) the conjugate of $a+\sqrt{b}$ is $a-\sqrt{b}$

Example; (i) Rationalize $\frac{a}{n\sqrt{b}}$ (ii) if $\sqrt{3}=1.732$, find the value of $\frac{2}{\sqrt{3}}$ correct to 3 significant figure (iii) Express $\frac{8-3\sqrt{6}}{2\sqrt{3}+3\sqrt{2}}$ in the form $m\sqrt{3}+n\sqrt{2}$ where m and n are rational numbers

(iv) Express $\frac{2}{(3\sqrt{5}-4)^2}$ in the form $a+b\sqrt{c}$, where a and b are rational numbers

Solution:

(i) $\frac{a}{n\sqrt{b}}=\frac{a\sqrt[n]{b^{n-1}}}{\sqrt[n]{b}\sqrt[n]{b^{n-1}}}=\frac{ab^{\frac{n-1}{n}}}{b^{\frac{1}{n}}.b^{\frac{n-1}{n}}}=\frac{a\sqrt[n]{b^{n-1}}}{b}$

(ii) $\frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{3}=\frac{2\times1.732}{3}=1.16$ correct to 3 sig.figure.

(iii) $\frac{8-3\sqrt{6}}{2\sqrt{3}+3\sqrt{3}}=\frac{8-3\sqrt{6}}{2\sqrt{3}+3\sqrt{2}}\times\frac{2\sqrt{3}-3\sqrt{2}}{2\sqrt{3}-3\sqrt{2}}=\frac{16\sqrt{3}-24\sqrt{2}-6\sqrt{18}+9\sqrt{12}}{12-18}$

$=\frac{16\sqrt{3}-24\sqrt{2}-18\sqrt{2}+18\sqrt{3}}{-6}=\frac{34\sqrt{3}-42\sqrt{2}}{-6}=-\frac{17}{3}\sqrt{3}+7\sqrt{2}$

$\Rightarrow m=-\frac{17}{3} \quad and \quad n=7$

(iv) $\frac{2}{(3\sqrt{5}-4)^2}=\frac{2}{45-24\sqrt{5}+16}=\frac{2}{61-24\sqrt{5}}=\frac{2(61+24\sqrt{5})}{(61-24\sqrt{5})(61+24\sqrt{5})}$

$\frac{122}{841}+\frac{48}{841}\sqrt{5},\Rightarrow a=\frac{122}{841},b=\frac{48}{841},c=5$

Equations involving surds

Example; (i) Solve the equation $\sqrt{(3x+1)}-\sqrt{(x+4)}=1$

(ii) simplify $\sqrt{5+2\sqrt{6}}$ (iii) Evaluate $\sqrt{9-4\sqrt{2}}$

Solution:

(i) $\sqrt{(3x+1)}-\sqrt{(x+4)}=1\equiv\sqrt{(3x+1)}=1+\sqrt{(x+4)}$ (1)

squaring both sides of (1), we have

$3x+1=[1+\sqrt{(x+4)}]^2\Rightarrow 3x+1=1+2\sqrt{(x+4)}+x+4$

$\Rightarrow 3x+1=x+5+2\sqrt{x+4}\Rightarrow 2x-4=2\sqrt{(x+4)}$ (2)

$\Rightarrow x-2=\sqrt{(x+4)}$

squaring both sides of (2) again yields

$(x-2)^2=x+4\Rightarrow x^2-5x=0\Rightarrow x=0 \quad or \quad x=5$

$if \quad x=0,\Rightarrow\sqrt{(3x+1)}-\sqrt{(x+4)}=-1(not\ solution)$

$if \quad x=5,\Rightarrow\sqrt{(3x+1)}-\sqrt{(x+4)}=1$

$\Rightarrow x=5 \quad is \quad the \quad solution$

(ii) Let $\sqrt{5+2\sqrt{6}}=\sqrt{x}+\sqrt{y}$ (1)

Squaring both sides of (1), we obtain

$5+2\sqrt{6}=x+y+2\sqrt{xy}$

$\Rightarrow 5=x+y, \quad 6=xy$

By inspection, $x=3, \quad y=2$

$$\Rightarrow \sqrt{5+2\sqrt{6}} = \sqrt{3}+\sqrt{2}$$

(iii) Let $\sqrt{9-4} = \sqrt{2 =}\sqrt{x-\sqrt{y}}$ (1)

The conjugate of (1) is

$$\sqrt{9+4}\sqrt{2} = \sqrt{x}+\sqrt{y} \quad (2)$$

Squaring both sides of (1), we have:

$$9-4\sqrt{2 =} -2\sqrt{xy}$$
$$\Rightarrow x+y=9 \quad (3)$$

Multiplying (1) and (2), we obtain

$$\sqrt{9x9-16.2} = x-y=7 \quad (4)$$

From (3) and (4)

$$\Rightarrow x+y=9$$
$$x-y=7$$
$$\therefore 2x=16$$
$$x=8, y=1$$

EXERCISES

(1) (i) $\sqrt{405}$ (ii) $\sqrt{98}$ (iii) $\sqrt{27 =} -\sqrt{12}$ (iv) $(\sqrt{7}-\sqrt{5})^2$ (v) $\dfrac{1}{\sqrt{3-1}}$ (vi) $\dfrac{3}{\sqrt{7-2}}$

(vii) $\dfrac{3}{\sqrt{7-2}}$ (vii) $\dfrac{\sqrt{3-1}}{\sqrt{3+1}}$ (viii) $\dfrac{2\sqrt{2+3}}{2\sqrt{2-1}}$

(2) If $a=2+\sqrt{3}$, Find the value of $a-\dfrac{1}{a}$

(3) Given $a\dfrac{1}{2-\sqrt{3}}$, $b\dfrac{1}{2+\sqrt{3}}$, find a^2+b^2

(4) Find the positive square roots of the following :
(i) $19+6\sqrt{2}$ (ii) $43+12\sqrt{7}$

(5) If $x=\dfrac{1}{2}(1-\sqrt{5})$, express $4x^3-3x$ in its simplest form.

INDICES

Definition 1: The product of a number with itself called the second power of the number, while the number, while its triple product is called third power of the number and its m factors product is called mth power of the number e.g. $axa = a^2$, $axaxa = a^3$, $axax...xm = a^m$

Definition 2: The number which expresses the power is called the index or the exponent of the power of a number e.g

The index of $a^2 = 2$
The index of $a^3 = 3$
The index of $a^m = m$

RULES OR LAWS OF INDICES

Given two positive integers m,n such that $m < n$.

(1) $a^m xa^n = a^{m+n}$

Since $a^m xa^n = (axax...xm)x(axax...xn)$

$[axax...x(m+n)] = a^{m+n}$

(2)

$= axaxax...(m-n)$

$= a^{m-n}$

LOGARITHMIC EQUATIONS

Example 1: Solve the equations

(i) $3x^2 = 9^{x+4}$

$\Rightarrow 3x^2 = (3^2)^{x+4}$

$3x^2 = 3^{2(x+4)}$

$\Rightarrow x^2 = 2(x+4)$

$\Rightarrow x^2 = 2n - 8 = 0$

$(x-4)(x+2) = 0$

$x - 4$ or $x = 2$

Example 2: Solve the equations

(ii) $3^{3x+1} = 5^{x+1}$

Taking the log; of both sides

$\Rightarrow \log_{10}\left(2^{3x+1} \right)\log_{10} 5^{x+1}$

$\Rightarrow 3x + 1Log_{10} 2 = x + 1Log_{10} 5$

$\Rightarrow \left(3Log_{10} - Log_{10} 5 \right)x = Log 5 - Log_{10} 2$

$\Rightarrow \left(3Log_{10} 8 - Log_{10} 5 \right)x = Log 5 - Log_{10} 2$

$\Rightarrow \left(Log_{10} 8 - Log_{10} 5 \right)x = Log 5 - Log_{10} 2$

$$\Rightarrow x = \frac{Log_{10} 5 - Log_{10} 2}{Log_{10} 8 - Log_{10} 5} = \frac{Log_{10} \frac{5}{2}}{Log_{10} \frac{8}{5}} - Log_{10} 2$$

$$\Rightarrow x = \frac{0.3979}{0.2041} = 1.95$$

(3) $(a^m)^n = a^{mn}$

$(a^m)^n = a^m x a^m x \ldots x n$

$(axax \ldots xm)x(axax \ldots xm) \ldots n$ times.

$axax \ldots xmn = a^{mn}$

(4) $\left(a\frac{1}{n} \right)^v = a$

$\left(a\frac{1}{n} \right) a\frac{1}{n} x a\frac{1}{n} x \ldots n$

$a\frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \ldots a$

Similarly, $a\frac{1}{n} = n\sqrt{a}$

$\left(a\frac{m}{n} \right)^n = a^m$

$a\frac{m}{n} = n\sqrt{a^m} = \left(n\sqrt{a^m} = \left(n\sqrt{a} \right)^m \right.$

(5) $a^0 = 1$, If $m = n$

$$a^{-n} = \frac{1}{a^n}$$

Examples: Evaluate (i) $(81)^{\frac{3}{4}}$ (ii) $(16)^{\frac{-5}{4}}$

Solution

(i) $(81)^{\frac{3}{4}}$ $= (81^3 = 4\sqrt{531441} = (3^4)^{\frac{3}{4}} = 3^3 = 27$

(ii) $(16)^{\frac{-5}{4}} = \dfrac{1}{(2^4)\frac{5}{4}} = \dfrac{1}{2} = \dfrac{1}{32}$

Exercises

(1) Show that $\sqrt{x} - \sqrt{a} = \dfrac{x-a}{\sqrt{x}+\sqrt{a}}$

(2) Evaluate (Godman).

LOGARITHMS

Definition: The logarithm of a tve no N to the base a is defined as the power of a which is equal to N, such that if

$$a^x = N$$
$$x = Log_a N$$

Since $a^1 = a \& a^0 = 1$

$\Rightarrow Log_a a = 1$ and $Log_a 1 = 0$

LAWS OF LOGARITHMS

(1) $\log_a(AB) = Log_a A + Log_a B$

(2) $\log_a \dfrac{A}{B} = Log_a A - Log_a B$

(3) $\log_a(A^B) = BLog_a A$

Example: Evaluate:

(i) $Log_3 9$

(ii) $Log_4 63$

(iii) $3^3 9 \Rightarrow Log_3 9 = 2$

(iv) $4^3 = 64, \Rightarrow Log_4 64 = 3$

Example: Use the table to evaluate:

(i) $Log_3 16 = \frac{Log_{10} 16}{Log_{10} 3} = \frac{1.2041}{0.4771} = 2.524$

Since from the transformation rule

$Log_9 N = Log_{10} b Log_6 N$

If $y = Log_b N, N = b^y$

$\Rightarrow Log_a N = Log(b^y) = y = b^y$

$Log_a N = Log_a b Log_b N$

If we put $N = a$ in $(*)$

$\Rightarrow Log_a a = Log_a b Log_b a = 1$

$$\Rightarrow Log_a b = \frac{1}{Log_a b} \qquad (**)$$

Another form of $(*)$ is $Log_a N = \frac{Log_b N}{Log_b a}$

Example: Show that:

$$Log_a (x^2 - x^2) = 2 + Log_a (1 - \frac{x^2}{a^2})$$

Solution:

$$Log_a (a^2 - x^2) = Log_a [\ a^2 + (1 - \frac{x^2}{a^2})\ ]$$

$$= Log_a a^2 + Log_a (1 - \frac{x^2}{a^2})\ ]$$

$$= 2 + Log_a (1 - \frac{x^2}{a^2})\ ]$$

COURSE CODE: *MTS 105*

COURSE TITLE: *Algebra and Trigonometry for Biological Sciences*

NUMBER OF UNITS: *3 Units*

COURSE DURATION: *Three hours per week*

COURSE DETAILS:

Course Coordinator: Drs I.A.Osinuga and I.O.Abiala

Email: osinugaia@unaab.edu.ng ; abialaio@unaab.edu.ng

Office Location: Departmental Office

Other Lecturers: Prof.Oguntuase;Drs Olajuwon, Adeniran, Akinleye and Mrs A**kwu**

COURSE CONTENT:

Matrix Algebra: Matrices. Algebra of Matrices. Determinant of a Matr**ix.** Properties. Inverse of a matrix. Solution of linear system of Equations.

COURSE REQUIREMENTS:

This is a compulsory course for all students in the University. In view of this, students are expected to participate in all the course activities and have minimum of 75% attendance to be able to write the final examination.

READING LIST:

1. Liadi, M.A. and Osinuga, I.A. A First Course in Mathematical Methods for Scientists and Engineers. Ibadan: Rasmed Publications (Nig.) Limited, 2011.
2. Tranter,C.J. and Lambe, C.G., Advanced Level Mathematics(Pure and Applied). St.Paul: The English Universities Press Limited, 1996.
3. Talbert, J.F. and Godman, A., Additional Mathematics for West Africa, United King: Longman Group Limited, 1984.

4. Akinguola, R.O. et al., Introductory Mathematics I(for Social and Management Sciences). Ago-Iwoye: CESAP Publication Unit,OOU. 1998.

MATRICES

Introduction

This chapter introduces an important branch of applied mathematics called matrices. The studies of matrices are of immense use because they provide short cut methods of calculations and this increases its suitability for practical problems.

We shall begin with the definition of matrices and related concept. We later discussed types of matrices and some operations on matrices and their applications in the solution of system of linear equations.

Definitions

Matrix is defined as a rectangular array of numbers or variables enclosed by a pair of bracket whose position in the matrix is significant. The individual entries are referred to as the elements of the matrix. The dimensions of a matrix (m x n) are defined as the numbers of rows m and columns n which is read as "m by n". If the number of rows equals the number of the columns in a matrix, such a matrix is referred to as a **square matrix**. In a **column vector** composed of a single column, such that the dimension is m x 1; if a matrix is a single row with dimension 1xn, it is a **row vector**.

Illustration 1:

Given A = $\begin{pmatrix} a_{11} & a_{12} & a_{13}\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23}\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3}\ldots & a_{mn} \end{pmatrix}$ and B = $\begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}$

Here A is a general array with any number m of rows and any number n of columns called m x n matrix.

Here B is a 2 x 3 matrix with b_{11} element as 2 and b_{13} as 3

Algebra of Matrices

Addition and Subtraction

The addition (subtraction) of matrices can only be carried out if the two matrices are of the same dimension. The sum of two m x n matrices A = a_{ik} and B = b_{ik} is the m x n matric C = c_{ik} with element $c_{ik} = a_{ik} + b_{ik}$ written as C = A + B.

Illustration 2:

If $A = \begin{pmatrix} 2 & 4 & 5 \\ 6 & 2 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \end{pmatrix}$

Then $A + B = \begin{pmatrix} 2 & 4 & 5 \\ 6 & 2 & 8 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 7 \\ 9 & 6 & 13 \end{pmatrix}$

Similarly,

$A - B = \begin{pmatrix} 2 & 4 & 5 \\ 6 & 2 & 8 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ 3 & -2 & 3 \end{pmatrix}$

Scalar Multiplication

Ordinary numbers such as 3, -2, 6, 4 etc. are called scalars. Multiplication of a matrix by a scalar involves multiplying each element of the matrix by the scalar.

Illustration 3:

Given k = 3 and $B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 4 \end{pmatrix}$

$kB = \begin{pmatrix} 3(1) & 3(2) & 3(1) \\ 3(2) & 3(2) & 3(4) \end{pmatrix} = \begin{pmatrix} 3 & 6 & 3 \\ 6 & 6 & 12 \end{pmatrix}$

Matrix Multiplication

Two matrices A and B are conformable for multiplication if the number of columns in A equals the number of rows in B. For instance, a 3 x 2 matrix can be multiplied with a 2 x 3 matrix since they are conformable (i.e. number of columns in the first equal number of rows in the second matrix) but a 2 x 3 cannot be multiplied with another 2 x 3 matrix.

Illustration 4:

Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \\ & \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 8 \\ 2 & 1 & 0 \end{pmatrix}$ calculate AB

1 5

Solution:

First test for the conformability of the two matrices. The number of columns in A equal the number of rows in B, 2 = 2; the matrices are conformable for multiplication; and the dimensions of the product matrix AB will be 3 x 3

$$AB = \begin{pmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 8 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2(1) + 3(2) & 2(2) + 3(1) & 2(8) + 3(0) \\ 1(1) + 2(2) & 1(2) + 2(1) & 1(8) + 2(0) \\ 1(1) + 5(2) & 1(2) + 5(1) & 1(8) + 5(0) \end{pmatrix}$$

$$\therefore \quad AB = \begin{pmatrix} 8 & 7 & 16 \\ 4 & 4 & 8 \\ 11 & 7 & 8 \end{pmatrix}$$

Illustration 5:

If A is the 3 x 2 matrix in illustration 4 above, find the product AC where

$$C = \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}$$

Solution:

The number of columns in A equals number of rows in C, 2 = 2; the matrices are conformable for multiplication. Hence the results is as follows:

$$AC = \begin{pmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 2(2) + 3(3) & 2(4) + 3(2) \\ 1(2) + 2(3) & 1(4) + 2(2) \\ 1(2) + 5(3) & 1(4) + 5(2) \end{pmatrix}$$

$$\therefore AC = \begin{pmatrix} 13 & 14 \\ 8 & 8 \\ 17 & 4 \end{pmatrix}$$

Laws in Matrix Algebra

It is important to note that some elementary laws of algebra also apply to matrices since their sum and difference is defined directly in terms of the addition and subtraction of their elements.

Also, matrices satisfy the associative and distributive laws of multiplication

Commutative Laws

Addition and subtraction of matrix are both commutative because it merely involves the summing of corresponding element of two matrices.

$A + B = B + A$

$A + (-B) = (-B) + A$ written as $A – B = -B + A$

Clearly, $AB \neq BA$ (i.e. matrix multiplication is non commutative) since the order of matrices representing two products are different. However, Scalar Multiplication is commutative (i.e. $kA = Ak$)

Note: $A – B \neq B – A$

Associative Laws

Matrix addition (subtraction) as stated earlier involves merely adding (subtracting) of corresponding element of two matrices and it does not matter in which sequence the matrices are added. For the same reason matrix addition is also associative.

$(A + B) + C = A + (B + C)$

The same is true of matrix subtraction if $A – B$ is written as $A + (-B)$. Moreover, if three or more matrices are conformable for multiplication, the associative law will apply as long as the matrices are multiplied in order of conformability. Thus,

$(AB)C = A (BC)$

Distributive Laws

Subject to the conditions of associative laws of multiplication, matrix multiplication is also distributive.

$A (B + C) = AB + AC$

provided that the products are defined.

Illustration 6:

Given $A = \begin{pmatrix} 2 & 3 \\ & \end{pmatrix}$ $B = \begin{pmatrix} 4 & 2 \\ & \end{pmatrix}$ verify (i) $A + B = B + A$

5 2 1 0 (ii) A – B = -B + A

Solution

(i) $A + B = \begin{pmatrix} 2 & 3 \\ 5 & 2 \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 6 & 2 \end{pmatrix}$

$B + A = \begin{pmatrix} 4 & 2 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 6 & 2 \end{pmatrix}$

$\therefore$ A + B = B + A i.e. commutative law of matrix addition holds

(ii) $A - B = \begin{pmatrix} 2 & 3 \\ 5 & 2 \end{pmatrix} - \begin{pmatrix} 4 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & 2 \end{pmatrix}$

$-B + A = \begin{pmatrix} -4 & -2 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & 2 \end{pmatrix}$

$\therefore$ A – B = -B + A

Illustration 7:

Given $A = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 1 & 6 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 1 \end{pmatrix}$ $C = \begin{pmatrix} 2 & 4 & 5 \\ 1 & 2 & 1 \end{pmatrix}$

Find (i) (AB)C (ii) A(BC)

Solution:

(i)$AB = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 1 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2(1) + 1(3) + 0(1) & 2(2) + 1(4) + 0(1) \\ 4(1) + 1(3) + 6(1) & 4(2) + 1(4) + 6(1) \end{pmatrix}$

$AB = \begin{pmatrix} 5 & 8 \\ 13 & 18 \end{pmatrix}$

$(AB)\,C = \begin{pmatrix} 5 & 8 \\ 13 & 18 \end{pmatrix} \begin{pmatrix} 2 & 4 & 5 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 18 & 36 & 33 \\ 44 & 88 & 83 \end{pmatrix}$

(ii) $BC = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 5 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 7 \\ 10 & 20 & 19 \end{pmatrix}$

$$A(BC) = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 1 & 6 \end{pmatrix} \begin{pmatrix} 4 & 8 & 7 \\ 10 & 20 & 19 \end{pmatrix} = \begin{pmatrix} 18 & 36 & 33 \\ 44 & 88 & 83 \end{pmatrix}$$

$\therefore$ Matrix Multiplication is associative

Types of Matrices

Null Matrices

Any (m x n) matrices with all its elements equal to zero is called a null matrix. As with ordinary number, addition or subtraction of null matrix to (from) a matrix leaves the original matrix unchanged and a null matrix is obtained if a matrix is multiplied by a null matrix.

Diagonal Matrices

A square (i.e. matrix with equal number of rows and columns) whose elements everywhere are zero except at the leading diagonal is called a diagonal matrix. An example is

$$A = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$$

Upper and Lower Triangular Matrices

A square matrix in which all the elements below the leading diagonal are zero is called the upper triangular matrix while the one in which all the elements above the leading diagonal are zero is called the lower triangular matrix. They are usually denoted by U and L respectively e.g.

$$U = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{pmatrix} \qquad L = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

Unit Matrices

Is a diagonal matrix with all its diagonal elements equal to unity and zero everywhere else. It is usually denoted by letter I. For example:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ is a 3 x 3 unit matrix.

Remarks: Clearly, there is similarity between the unit matrix I and the number 1. As with ordinary number where a number is multiplied by one equals itself so with unit matrices.

A matrix multiplied by the unit matrix equal itself provided the product is defined, i.e. AI = IA = A.

For example,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Multiplication of the unit matrix by itself leaves the matrix unchanged:

$I = I^2 = I^3 = I^4 = \ldots$

Transposed Matrix

The resulting matrix obtained by interchanging rows and columns of a matrix ix called a transposed matrix. Supposed A is given by

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 4 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$$

Then the transpose of A is given by

$$A^T = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$$

Symmetric and Skew – Symmetric Matrices

A real square matrix $A = a_{jk}$ is said to be symmetric if it is equal to its transpose

$A^T = A$ i.e. $a_{jk} = a_{kj}$ $(j, k = 1,\ldots, n)$

Whereas a real square matrix $A = a_{jk}$ is said to be skew – symmetric if

$A^T = -A$ i.e. $a_{jk} = -a_{jk}$ $(j, k = 1,\ldots, n)$

Determinant of a Matrix

The determinant of a matrix A is a number or scalar defined only for square matrices and usually denoted by |A| or det (A). It is calculated from the products of the elements of the matrix. If the determinant of a matrix equal zero, the matrix is termed singular; otherwise it is said to be non-singular.

The determinant |A| of a 2 x 2 matrix, called a **second-order determinant** is defined as follows:

$$\text{Let } A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad |A| = ad - cb$$

Similarly, for a 3 x 3 matrix, the determinant called **third-order determinant** is calculated thus:

$$A = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

$$|A| = a \begin{vmatrix} e & h \\ f & i \end{vmatrix} + d(-1) \begin{vmatrix} b & h \\ c & i \end{vmatrix} + g \begin{vmatrix} b & e \\ c & f \end{vmatrix}$$

Properties of a determinant

(1) The determinant of a square matrix is zero if every element of a row or column of such matrix is zero.

(2) The determinant of a matrix equals the determinant of its transpose: $|A| = |A^T|$

(3) If every elements of any row (or column) are multiplied by a constant, the value of the determinant is multiplied by this constant: |kA| = k|A|.

(4) The value of a determinant is unaffected by interchanging the elements of all corresponding rows and columns.

(5) The value of a determinant is zero if any two of its rows (or columns) are proportional.

(6) The value of a determinant is unchanged if any non-zero multiple of one row (or column) are added (or subtracted) to the corresponding elements of any other rows (or columns).

Illustration 8:

$$\text{Given } A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 4 \\ 8 & 9 & 0 \end{pmatrix}$$

Find (i) Determinant of A

(ii) Determinant of the Transpose of A

and indicate which property of the determinant this illustrate.

Solution:

(i) $|A| = 1(-36) - 2(-32) + 1(11) = -36 + 64 + 11 = 39$

(ii) $A^T = \begin{pmatrix} 1 & 3 & 8 \\ 2 & 2 & 9 \\ 1 & 4 & 0 \end{pmatrix}$ $|A^T| = 1(-36) - 3(-9) + 8(6) = 39$

Since $|A| = |A^T|$, this illustrates properties 2

Minor, Cofactors and Adjoint of a Matrix

Given a determinant of order 3 written as a square array of 9 quantities enclosed between vertical bars.

$$D = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

By deleting the ith row and kth column from the determinant D, we obtain a sub-determinant of the matrix called a **minor**. Thus a **minor** $|m_{ik}|$ is the determinant of the sub-matrix obtained by deleting the ith row and kth column of the matrix.

Using a (3 x 3) matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Deleting the first row and first column in A, i.e.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{we obtain } |M_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Similarly deleting first row and second column

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & & \\ & & \end{pmatrix} \quad \begin{vmatrix} & \\ & \end{vmatrix}$$

$$\begin{matrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \quad \text{we obtain } |M_{12}| = \begin{matrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

In the same manner we have

$$|M_{13}| = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

A **co-factor** $|C_{ik}|$ is a minor multiplied by $(-1)^{i+k}$: thus

$$|C_{ik}| = (-1)^{i+k} |M_{ik}|$$

i.e. co-factor is a matrix in which every entries a_{ik} is replaced with its cofactor $|C_{ik}|$. An **adjoint** matrix is the transpose of a cofactor matrix.

Thus ifA = $\begin{pmatrix} |C_{11}| & |C_{12}| & |C_{13}| \\ |C_{21}| & |C_{22}| & |C_{23}| \\ |C_{31}| & |C_{32}| & |C_{33}| \end{pmatrix}$

Adjoint A = $C^T = \begin{pmatrix} |C_{11}| & |C_{21}| & |C_{31}| \\ |C_{12}| & |C_{22}| & |C_{32}| \\ |C_{13}| & |C_{23}| & |C_{33}| \end{pmatrix}$

Illustration 9:

Given A = $\begin{pmatrix} 3 & -4 & -6 \\ 1 & 2 & -4 \\ 2 & -5 & 2 \end{pmatrix}$ find (i) The cofactor of matrix A

(ii) Adjoint of A

$$\begin{pmatrix} \begin{vmatrix} 2 & -4 \\ -5 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -4 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix} \\ -\begin{vmatrix} -4 & -6 \\ -5 & 2 \end{vmatrix} & \begin{vmatrix} 3 & -6 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & -4 \\ 2 & -5 \end{vmatrix} \\ & & \begin{vmatrix} & \\ & \end{vmatrix} \end{pmatrix}$$

$$\begin{vmatrix} -4 & -6 \\ 2 & -4 \end{vmatrix} \quad - \quad \begin{vmatrix} 3 & -6 \\ 1 & -4 \end{vmatrix} \quad \begin{matrix} 3 & -4 \\ 1 & 2 \end{matrix}$$

$$\text{Cof.A} = \begin{pmatrix} -16 & -10 & -9 \\ 38 & 18 & 7 \\ 28 & 6 & 10 \end{pmatrix}$$

(ii) Adjoint A = $(\text{Cof. A})^T$ $= \begin{pmatrix} -16 & 38 & 28 \\ -10 & 18 & 6 \\ -9 & 7 & 10 \end{pmatrix}$

Inverse of a Matrix

In the matrix algebra, the inverse of a matrix A is denoted by A^{-1} and calculated only for square, non-singular matrix such that

$$A^{-1} A = I = AA^{-1}$$

It can easily be verified that multiplication of matrices and their inverses is commutative and that multiplying a matrix by its inverse yields an identity matrix. Thus, the inverse of a matrix has the property as the reciprocal in ordinary algebra.

In this section, we present a method of calculating the inverse of a matrix:

Adjoint method: The formula for finding the inverse of a given matrix A is given as:

A^{-1}= adjoint A / det. A

where det. A is the determinant of A.

Illustration 10:

Find the inverse of a matrix A = $\begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix}$

Solution:

|A| = -4 – 15 = -19

$$\text{Cof. } A = \begin{pmatrix} -2 & -5 \\ -3 & 2 \end{pmatrix} \quad \text{then}$$

$$(\text{Cof. } A)^T = \begin{pmatrix} -2 & -3 \\ -5 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{(\text{Cof. } A)^T}{|A|} = -1/19 \begin{pmatrix} -2 & -3 \\ -5 & 2 \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} 2/19 & 3/19 \\ 5/19 & -2/19 \end{pmatrix}$$

6.9 Solution of Systems of Linear Equations Using Matrix Inverse

Consider the following system of linear equation in unknowns $x_1, x_2, \ldots, x_n$:

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_2 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad \vdots \qquad\qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{array}$$

where the coefficient a_{ix} and b_i ($i, x = 1, 2, 3, \ldots \ldots, n$) are known constants. This equation can be written in the matrix form as:

$$\begin{pmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \ldots & \ldots \quad \ldots \\ \ldots & \ldots \quad \ldots \\ \ldots & \ldots \quad \ldots \\ a_{n1} & a_{n2} \ldots a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ \ldots \\ \ldots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ \ldots \\ \ldots \\ b_n \end{pmatrix} \quad \text{or } AX = B$$

Assuming $B \neq 0$ and A to be non-singular matrix (i.e. A^{-1} exists).

Multiplying both sides of the equation by A^{-1} gives

$$A^{-1} AX = A^{-1}B \quad (A^{-1} A = I)$$

$$\therefore X = A^{-1}B$$

The solution of the equation AX = B is given by the equation $X = A^{-1}B$ i.e. the product of the inverse of the coefficient matrix A^{-1} and the column vector of constants B.

Illustration 11:

Solve the equation $4x + 5y = 0$

$2x + 3y = 0$

Solution: First write the equation in matrix form

$$AX = B$$

$$\begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

In order to find the inverse of of the coefficient matrix, proceed as follows:

$$A = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}, \qquad |A| = 2$$

$$\text{Cof. } A = \begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix} \qquad (\text{Cof. } A)^T = \begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix}$$

$$A^{-1} = \frac{\text{Adjoint } A}{|A|} = \frac{(\text{Cof. } A)^T}{|A|} = ½ \begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix}$$

$$\text{Thus } X = A^{-1} B = \begin{pmatrix} 3/2 & -5/2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

So the solution of the equation is x = 3, y = -2.

Cramer's Rule

Cramer's rule is rule that provides a shorthand method of solving linear equation. The concept follows that if $D \neq 0$, the system of linear equation has the unique solution given by:

$$X_i = \frac{D_i}{D}$$

Where D_1 is the determinant of a special matrix formed from the original matrix by replacing the column of coefficient of x_i with the column vector of constants, D is the determinant of the coefficient matrix and x_i is the ith unknown variable in a series of equation.

Illustration 12:

Solve by Cramer's rule

$$2x_1 + 3x_2 = 8$$

$5x_1 - 2x_2 = 1$

Solution: Express the equations in matrix form:

$$\begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$$

The determinant of matrix A is

$$D = \begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix} = -4 - 15 = -19$$

To solve for $x_1 = D_1/ D$, replace a column 1, the coefficient of x_1, with vector of constants B, forming

$$D_1 = \begin{vmatrix} 8 & 3 \\ 1 & -2 \end{vmatrix} = -16 - 3 = -19$$

And $x_1 = \frac{D_1}{D} = \frac{-19}{-19} = 1$

Similarly, to solve for $x_2 = D_2 / D$, replace column 2, the coefficient of x_2, with vector of constants B, forming:

$$D_2 = \begin{vmatrix} 2 & 8 \\ 5 & 1 \end{vmatrix} = 2 - 40 = -38$$

Which gives $x_2 = \frac{D_2}{D} = \frac{-38}{19} = 2$

$\therefore$ $x_1 = 1$ and $x_2 = 2$

Remarks: Given systems of the three linear equations of the matrix form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

By Cramer's rule, the system of three linear equation yields the solution

$x_1 = \frac{D_1}{D}$ $\quad$ $x_2 = \frac{D_2}{D}$ $\quad$ and $x_3 = \frac{D_3}{D}$

Where $D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ & & \end{vmatrix}$ $\quad$ $D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ & & \end{vmatrix}$

$$\begin{matrix} b_3 & a_{32} & a_{33} \end{matrix} \qquad \begin{matrix} a_{31} & b_3 & a_{33} \end{matrix}$$

and $D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$

Illustration 13:

Solve by Cramer's rule

$-x + 3y – 2z = 7$

$3x + 3y = -3$

$2x + y + 2z = -1$

Solution: Rewrite the equations as:

$$\begin{vmatrix} -1 & 3 & -2 \\ 3 & 0 & 3 \\ 2 & 1 & 2 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} 7 \\ -3 \\ -1 \end{vmatrix}$$

The find the determinant D,

$$D = \begin{vmatrix} -1 & 3 & -2 \\ 3 & 0 & 3 \\ 2 & 1 & 2 \end{vmatrix} = -3$$

Also,

$$D_1 = \begin{vmatrix} 7 & 3 & -2 \\ -3 & 0 & 3 \\ -1 & 1 & 2 \end{vmatrix} = -6, \quad D_2 = \begin{vmatrix} -1 & 7 & -2 \\ 3 & -3 & 3 \\ 2 & -1 & 2 \end{vmatrix} = -3 \text{ and}$$

$$D_3 = \begin{vmatrix} -1 & 3 & 7 \\ 3 & 0 & 3 \\ 2 & 1 & -1 \end{vmatrix} = 9$$

Solving for x_1, x_2 and x_3; we have

$x_1 = D_1/D = -6/-3 = 2$; $x_2 = D_2/D = -3/-3 = 1$ and $x_3 = D_3/D = 9/-3 = -3$

$\therefore\ x_1 = 2$, $x_2 = 1$ and $x_3 = -3$

Exercises:

COURSE CODE: *MTS 101*

COURSE TITLE: *Algebra*

NUMBER OF UNITS: *3 Units*

COURSE DURATION: *Three hours per week*

COURSE DETAILS:

Course Coordinator: Dr. I.A.Osinuga and Miss A.D.Akinola

Email: osinugaia@unaab.edu.ng ; akinolaad@unaab.edu.ng

Office Location: Departmental Office

Other Lecturers: Prof.Oguntuase;DrsAdeniran,Akinleye,Omeike;Mr.Ilodje and Mrs A**kwu**

COURSE CONTENT:

Sequences and Series: AP and GP, Means, nth term and limits. Binomial Theorem for any index. B inomial series.

COURSE REQUIREMENTS:

This is a compulsory course for all students in the University. In view of this, students are expected to participate in all the course activities and have minimum of 75% attendance to be able to write the final examination.

READING LIST:

5. Tranter,C.J. and Lambe, C.G., Advanced Level Mathematics(Pure and Applied). St.Paul: The English Universities Press Limited, 1996.
6. Talbert, J.F. and Godman, A., Additional Mathematics for West Africa,
7. Akinguola, R.O. et al., Introductory Mathematics I(for Social and Management Sciences). Ago-Iwoye: CESAP Publication Unit,OOU. 1998.

SEQUENCES AND SERIES

Sequences

Sequences can be defined as a succession of things or number occurring in some particular order that can be determined by a definite law. For example, the sets (i) 1, 3, 5, 7, 9, …

(ii) 2, 4, 6, 8, 10, …

(iii) 1, 4, 9, 16, 25, …

(iv) 4, -1, -6, -11, …

are all sequences and each number is a term of the sequence. The dots … indicate that the sequence continues indefinitely.

Series

Series on the other hand can be described as the resulting expression when the terms of a sequence are linked together with signs of subtraction or addition.

Examples of series include:

(i) $1 + 3 + 5 + 7 + 9 + \dots$

(ii) $1 + 4 + 9 + 16 + 25+ \dots$

(iii) $4 - 1 - 6 - 1 - \dots.$

Observe that when the terms of a finite sequence are added together, we obtain a finite series. Thus, the series (ii) contains a finite sequence of 5 terms to give a finite series but series (i) and (iii) continues infinitely (i.e. infinite series).

The nth Term

It is possible to give a formula for the nth term of a sequence. More so, it is easy to write down any term in the sequence once the nth term or general term is known. The general term are usually denoted by T_n or U_n. We shall illustrate with examples how to write down the nth term formula of some of the above sequences.

Illustration 1:

Write down the nth term formula for the sequences (i) and (iv) above.

Solution: The first sequence is 1, 3, 5, 7, 9 …

We construct a table in order to see the pattern of the sequence

Table 1

Number of term	1st	2nd	3rd	4th	5th	….	10th ….	nth
Term	1	1+2	1+2+2	1+2+2+2	1+2+2+2+2		…	…
		1+2(1)	1+2(2)	1 + 2(3)	1 + 2(4)		1 + 2(9)	1+2(n-1)

Hence, the nth term of the sequence is given by:

$$1 + 2(n - 1) = 1 + 2n - 2 = 2n - 1.$$

The sequence (iv) is 4, -1, -6, -11, …

Table 2

No of term	1st	2nd	3rd	4th …	10th …	nth
Term	4	4-5	4-5-5	4-5-5-5	…	…
		4-5(1)	4-5(2)	4-5(3)	4-5(9)	4-5(n-1

$\therefore$ The nth term is $4 - 5(n - 1) = 4 - 5n + 5 = 9 - 5n$

Illustration 2:

Write out the first 4 terms of the sequence whose nth term is (i) n^2 (ii) 2^{n-1}

Solution:

(i) Given nth term to be n^2; Then

1^{st} term becomes	1^2	=	1
2^{nd} term becomes	2^2	=	4
3^{rd} term becomes	3^2	=	9
4^{th} term becomes	4^2	=	16

$\therefore$ The first 4 terms are 1, 4, 9, 16

Note that the nth term formula $\mathbf{n^2}$ is general term of the sequence (iii) given above

(ii) Given nth term to be 2^{n-1}. Then

1^{st} term gives	$2^{1-1} = 2^0 = 1$
2^{nd} term gives	$2^{2-1} = 2^1 = 2$
3^{rd} term gives	$2^{3-1} = 2^2 = 4$
4^{th} term gives	$2^{4-1} = 2^3 = 8$

$\therefore$ The first four terms are 1, 2, 4, 8

Progressions

If a sequence of terms is governed by a definite law, the terms are said to form a progressions. Among the various forms of progressions, we have includes Arithmetic, Geometric and Harmonic progressions.

We shall however lay emphasis on arithmetic and geometric progression

Arithmetic Progression (AP)

In an Arithmetic Progression, the sequences of terms differ from one another by a constant quantity which may be positive or negative. This constant quantity is called the common difference (d). For example, the sets

(i) 2, 4, 6 8, 10, …

(ii) 5, 9, 13, 17, 21, …

(iii) 4, -1, -6, -11, …

are all arithmetic progressions or arithmetic sequences. Thus, the common differences (d) in the three sequences given above are 2, 4, and -5 respectively.

Generally, if we denote the first term in AP by 'a' the nth term of an AP is denoted by T_n given by:

$$\mathbf{T_n = a + (n - 1)\, d}$$

Illustration 3: Given the sequence 2, 5, 8, 11, …

Find (i) the formula for the general term

(ii) the 28^{th} term of the sequence

Solution:

(i) $a = 2, d = 3$

$T_n = a + (n - 1)\, d = 2 + (n - 1)\, 3 = 3n - 1$

(ii) $T_{28} = a + (28 - 1)d = a + 27d = 2 + 27(3) = 83$

Illustration 4:

The 5th term of an AP is -10 and the 10^{th} term is -25. What is the sequence?

Solution: $T_5 = a + 4d = -10$ … (1)

$T_{10} = a + 9d = -25$ … (2)

Subtracting (2) from (1)

$-5d = 15$ i.e. $d = -3$

Then from (1) $a - 12 = -10$ i.e. $a = 2$

Hence the AP is 2, -1,-4, -7, …

Illustration 5:

How many terms has the AP, if the first and the last terms of an AP with d = 2 are 0 and 10 respectively?

Solution:

$a = 0, T_n = 10, d = 2$

$T_n = a + (n - 1)d$

$10 = 0 + (n - 1)\, 2$ i.e. $10 = 2n - 2$

or $n = 6$

$\therefore$ There are 6 terms of the sequences.

The Arithmetic Mean

In an AP, if the three consecutive terms are given as x, y, z. Then the common difference $d = y - x = z - y$

i.e. $y + y = z + x$

or $y = \dfrac{z + x}{2}$

Therefore, y is called the arithmetic mean of x and z. The arithmetic mean of 5 and 17 is given by $5 + 17 / 2 = 11$

Illustration 6: Insert 2 arithmetic mean between 2 and 11

Solution: To do this, find 2 numbers x, y between 2 and 11 such that 2, x, y, 11 forms an AP

$a = 2, a + 3d = 11, d = 3$

The required numbers are 2, 5, 8, 11

The Sum of an AP

The nth term of an AP whose first term is a and common difference d has been given as $T_n = a + (n - 1)d$

Then $T_2 = a + d$

$T_3 = a + 2d$

$T_4 = a + 3d$ etc.

If the sum of the 1st n terms of an AP is denoted by S_n, we have

$S_n = a + a + d + a + 2d + a + 3d + \dots + a + (n - 1) d \quad \dots \quad (1)$

Write the series backward and it becomes

$S_n = a + (n - 1) d + \dots + a + 3d + a + 2d + a + d + a \quad \dots \quad (2)$

Adding each term of the series in (1) to the corresponding term of series (2) we have

$2S_n = n [2a + (n - 1) d]$

or $S_n = {}^n/_2 [2a + (n - 1) d]$

Illustration 7:

Find the sum of the first 20 terms of AP: $0 + 2 + 4 + 6 + 8 + 10 + \dots$

Solution: $Sn = {}^n/_2 [2a + (n - 1) d]$

$= {}^{20}/_2 [0 + (20 - 1) 2] = 10 (38) = 380$

Exercise 1:

1. The given expression is the nth of a sequence. Write down the first six terms of each sequence. (i) n (b) $\frac{n + 1}{2}$ (iii) 2^{n-2} (iv) $5 – 2n$ (v) $(-1)^n$

(2) Find the possible expression for T_n for each of the following sequences and write down the next three terms

(i) 1, ½, ⅓, ¼, $^1/_5$, …

(ii) 2, 4, 6, 8, 10, …

(iii) 3, 9, 27, …

(3) Two sequences are defined by $T_n = n^2 – n + 4$ and $V_n = 2^{n-1}$. Find for each sequence which term is equal to 16.

(4) In a sequence given by $Tn = a + bn$, the 7^{th} and 15^{th} terms are 19 and 42 respectively. Find the values of a and b.

(5) Tn is given for each sequence (i) $3n+1$ (ii) $n(n + 2)$ (iii) $2n^3$. Find T_3 and T_{10}

(6) Find the 23rd term of the AP: 3, -1, -5, -9, …

(7) If (i) $a = 3$, $T_{12} = -41$ find d

(ii) $a = 2$, $d = 3$, $T_n = 59$ find n

(iii) $T_{10} = 18$, $d = 2$, find a

(8) The 5^{th} and 10^{th} term of an AP are 8 and 18 respectively. Find the AP and its 15^{th} term.

(9) If 1, x, y, 19 are in AP; find x and y

(10) Insert 2 arithmetic mean between 3 and 18

(11) Find the arithmetic mean of

(i) -7 and 3

(ii) 3 and 13

(iii) 12 and 4

(12) Find the sum of 12 terms of the series 7 + 13 + 19 + …

(13) How many terms of the AP 0 + 2 + 4 + 6 + … make a total of 380?

(14) What is the nth terms of the AP: 10, 6, 2, -2, …

(15) How many terms of the AP: 3, 7, 11, … must be added together to produce a total of 300?

Geometric Progressions (GP)

In a geometric progression, each term in a sequence of terms is a constant multiple of the preceding one. This constant multiple is referred to as the common ratio (r). Thus, for example

(i) 2, 4, 8, 16, …

(ii) 2, 1, ½, ¼, …

(iii) -3, 9, -27, 81, …

are geometric progressions with common ratio 2, ½ and -3 respectively.

If a is the first term of a GP and r is the common ratio, the sequence will be

$T_1 = a$
$T_2 = ar$
$T_3 = ar^2$
$T_4 = ar^3$
.
.
.
$T_{10} = ar^9$
.
.
.
$T_n = ar^{n-1}$

Following this pattern, we observe that the nth term of a GP is

$$\mathbf{Tn = ar^{n-1}}$$

Illustration 8:

Given the sequence -2, -4, -8, -16, …

Find (i) the 9th term of the above sequence

(ii) a formula for the nth term.

Solution:

(i) Hence a = -2 and r = 4/-2 = -8/4 = -2

$T_9 = ar^8 = (-2)\ (-2)^8 = -512$

(ii) $T_n = (-2)\ (-2)^{n-1} = (-2)^n$

Illustration 9:

Find the nth term of the series 4 + 6 + 9 + 27/2 + …

Solution: In the above series $a = 4$, $r = {}^3/_2$

$T_1 = a = 4$

$T_2 = ar = 4\,({}^3/_2) = 6$

$T_3 = ar^2 = 4\,({}^3/_2)^2 = 0$

.

.

.

$T_n = ar^{n-1} = 4\,({}^3/_2)^{n-1}$

$\therefore$ The nth term of the series is $4\,({}^3/_2)^{n-1}$

Illustration 10:

The second term of a GP is 12 and the 9th term is 108. Find the two possible value of a (first term) and r.

Solution: $T_2 = ar = 12$ and $T_4 = ar^3 = 108$

Then $\dfrac{T_4}{T_2} = \dfrac{ar^3}{ar} = \dfrac{108}{12} = 9$ or $r^2 = 0$ i.e $= \pm 3$

From the first equation, when $r = 3$, $ar = 12$ or $3a = 12$ i.e. $a = 4$

Also, when $r = -3$, we have $ar = 12$ or $-3a = 12$ i.e. $a = -4$

$\therefore$ The GP is either 4, 12, 36, 108, … or -4, 12, -36, 108, …

The Geometric Mean

The Geometric Mean follows the same procedure as in arithmetic mean. Given three consecutive terms x, y, z; the common ratio is

$$\frac{y}{x} = \frac{z}{y} \text{ or } y2 = xz \text{ i.e. } y = \sqrt{xz}$$

which is the condition for the three consecutive terms x, y, z to form a GP. The geometric mean of two number is the square root of their products e.g. the geometric mean of 3 and 27 is $\sqrt{3} \times 27 = \sqrt{81} = 9$

Illustration 11: Insert 2 GM between 1 and 125

Solution: We find numbers x, y such that 1, x, y, 125 forms a GP. The terms are a, ar, ar^2 and ar^3.

$a = 1$, $ar^3 = 125$ or $r = 5$

Hence $x = ar = 5$, $y = ar^2 = 25$

$\therefore$ The required Geometric Mean are 5, 25

The Sum of a GP

Generally, let S_n be the sum of n terms of the GP with first term a and common ratio r. Since the nth term is ar^{n-1}, so that

$$S_n = a + ar + ar^2 + \dots + ar^{n-1} \qquad \dots \qquad (1)$$

Multiply (1) by r, we have

$$rS_n = ar + ar^2 + ar^3 + \dots + ar^n \qquad \dots \qquad (2)$$

Subtracting (2) from (1) becomes

$$S_n - rS_n = a - ar^n$$

$$S_n (1 - r) = a (1 - r^n) \text{ or } S_n (r - 1) = a (r^n - 1)$$

Therefore,

$$S_n = \frac{a (r^n - 1)}{r - 1} = \frac{a (1 - r^n)}{1 - r}, \ r \neq 1$$

The first form is more suitable if $r > 1$, the second form when $r < 1$. If $r = 1$, the formula cannot be used. In this case however, the sum of the series is na.

Illustration 12: Find the sum of the first 6 terms of the GP 1, 3, 9, 27, …

Solution: In the GP, $a = 1$, $r = 3$

$$S_6 = \frac{a (r^n - 1)}{r - 1}, \ r > 1$$

$$S_6 = \frac{1 [(3)^6 - 1]}{3 - 1} = \frac{3^6 - 1}{2} = \frac{728}{2} = 364$$

Remarks:

Observe that when r is very small (i.e. approaching zero) and n is sufficiently large the sum of GP to infinity of a GP is

$$S = \frac{a}{1 - r}$$

Since the limit of $r^n \rightarrow 0$ and $n \rightarrow \infty$

Exercise 2:

Write down the term indicated in each of the following Geometric Progression

1. 6, 3/2, 3/8, …, 7th
2. 4, 12, 36, … , 15th
3. -3, 9, -27, …, 6th
4. 5, -5/2, 5/4, … , 5th
5. 9, -3, 1, … , 10th

What is the nth term of the following GP

6. 2, 4, 8, …
7. 6, 18, 54, …
8. y, 2yz, $4yz^2$, …
9. 1000, 100, 10, …
10. 28, -14, 7, …
11. The 2nd and 6th terms of a GP (r > 0) are 4 and 64 respectively. Find the sum of first 5 terms.
12. If 1, x, y, 27 are in GP, find x and y
13. Find the geometric mean between x and $4xy^2$
14. How many terms of GP 4 + 9 + … must be taken to make the sum exceed 5000?
15. Find the sum to infinity of the series 75 + 45 + 27 + …

Applications

Compound Interest

When money is saved in a bank, the interest is paid on this money saved with compound interest. The interest is added to the principal at the end of each interval so that the successive terms of the series forms a geometric progression.

In this case, the principal and the interest for each interval increases.

If the sum of money, P is invested for M years at K% per annum compound interest, the amount S, after M years is given by the formula

$$\mathbf{S = P\ (1 + K)^{M}}$$

Illustration 13:

A trader saved ₦1,000 in a bank which pay interest at 4% per annum. Find the amount in the trader's savings account after a year, 2 years, 3 years, …, 6 years.

Solution: The Compounding Formula is $S = P (1 + K)^M$

After 1 year $S = 1000 (1 + 0.04)^1$ = ₦1040.00k

After 2 years $S = 1000 (1 + 0.04)^2$ = ₦1081.60k

After 3 years $S = 1000 (1 + 0.04)^3$ = ₦1124.86k

After 6 years $S = 1000 (1 + 0.04)^6$ = ₦1265.32k

Using Tabular Form:

P for 1st year = 1000;

I for 1st year = 40;

Principal year = 1040 amount after 1st year

Interest for 2nd year	41.6	
Principal for 3rd year	1081.60k	amount after 2nd year
Interest for 3rd year	43.264	
Principal for 4th year	1124.864	amount after 3rd year
Interest for 4th year	44.99	
Principal for 5th year	1169.854	amount after 4th year
Interest for 5th year	46.794	
Principal for 6th year	1216.65	amount after 5th year
Interest for 6th year	48.66	
	1265.32	amount after 6th year

When the present value is required given the future value the compounding formula, is restated in terms of **discounting** to a present value as

$$\mathbf{P = \frac{S}{(I + K)^M}}$$

Illustration 14:

How much will have to be invested now to produce N862 after 6 years with a 9½ compound interest rate?

$$P = \frac{862}{(1 + {}^{19}/_{200})^6} = 500$$

Exercise 3:

1. Find the amount that N5,000 becomes if saved for 5 years at 6% per annum compound interest
2. A company invests N10,000 each year for 5 years at 9% compound interest. What will be the fund be after 5 years?
3. How much will have to be invested now to produce N20,000 after 10 years with a 10% Compound Interest Rate?
4. A firm rents his premises and the rented agreement provided for a regular annual increase of N2650. If the rent in the first year is N8500. What is the rent in the tenth year?
5. A man borrows N9000 to buy a house at 8% per annum compound interest. He repays N920 at the end of that year. How much does he still owe at the end of 3 years?
6. Find the sum of the first 20 terms of the AP 5, 11, 17, 23, …
7. 16 cows were slaughtered for sale in a market town during the first week of January 197 and four more cows during each subsequent week. Assuming thee are 4 weeks in each month, determine the number of cows slaughtered for sale at the end of December of the same year?
8. How much will N10,000 amount to at 8% per annum compound interest over 15 years?
9. (i) How much will N20,000 amount to at 3% per annum compound interest over 10 years
   (ii) What compound rate of interest will be required to produce N5,000 after five years with an initial investment of N4,000.
10. The profit of a supermarket shows an annual increase of 6%. Assuming the current market trends continue, what will be the supermarkets annual profit in the 4th year if the first profit was N20,000? Find also the total profit for the first 4 years.

BINOMIAL SERIES

Permutation: A permutation is an arrangement of a number of objects in a particular order.

Combination: When a selection of objects is made with no regard being paid to order, it is referred to as combination e.g. ABC, BAC, CAB are different permutation but the same combination of letters.

Factorial Notation

$$n! = n(n-1)(n-2)(n-3)\ldots 3.2.1$$

e.g. $6! = 6.5.4.3.2.1$

$4! = 4.3.2.1$

Notations:

1. Permutation: ${}^nP_r = \frac{n!}{(n-r)!}$

2. Combination: ${}^nC_r = \frac{n!}{(n-r)!\,r!}$

Remarks:

(a) ${}^nC_1 = {}^nC_{n-1}$

(b) ${}^{n+1}C_{r+1} = {}^nC_r + {}^nC_{r+1}$

(c) $\frac{{}^nP_r}{{}^nC_r} = r!$ (The proof of the above is left as exercise for the reader)

Examples:

$${}^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5.4.3.2.1}{2.1} = 60$$

$${}^5C_3 = \frac{5!}{(5-3)!\,3!} = \frac{5!}{2!3!} = \frac{5.4.3.2.1}{2.1.3.2.1} = 10$$

2.2 BINOMIAL THEOREM

Observe that $(a + b)^2 = a^2 + 2ab + b^2$. It might be difficult for some of us to expand $(a + b)^3$ and other higher powers of $(a + b)$ without doing some serious work on paper; through long multiplication.

But since $a^0 = 1$

$(a + b)^0 = 1$

$$(a + b)^1 = a + b$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \text{ etc.}$$

If the coefficients are written alone, we have

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array} \qquad \text{etc.}$$

On studying the order closely, we observe that in each case, the first and last coefficients are unity.

Furthermore, we noticed that each of the other coefficient $(a + b)^{n+1}$ is the sum of the corresponding coefficient, and their preceding one in the expansion of $(a + b)^n$. Thus, we can lay out coefficients for the successive powers in the form of a triangle called **Pascal's Triangle.**

However, for large values of n, it might not be easy to use Pascal's Triangle, thus to write down the expansion of $(a + b)^n$ for large n, we use a formular for the coefficient of $a^{n-r}b^r$ in the expansion of $(a + b)^n$. We shall denote this by the symbol n which alternatively leads to **Binomial Theorem.**

$(a + b)^n = {}^nC_0\, a^nb^0 + {}^nC_1\, a^{n-1}\, b^1 + {}^nC_2\, a^{n-2}\, b^2 + \ldots + {}^nC_n\, a^0\, b^n$

Illustration 1:

(1) Expand $(x + y)^6$ in descending power of n

Solution: There will be seven terms involving

$x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6$

Their coefficients obtained from Pascal's triangle are respectively,

1 16 15 20 15 6 1

Thus, the required expansion is

$$x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6$$

Similarly, we could verify that

$(x + y)^6 = {}^6C_0x^6y^0 + {}^6C_1x^5y + {}^6C_2x^4y^2 + {}^6C_3x^3y^3 + {}^6C_4x^2y^4 + {}^6C_5xy^5 + {}^6C_6x^0y^6$

$= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

Recall that:

${}^nC_0 = 1$

${}^nC_1 = n, \qquad {}^nC_2 = \dfrac{n\,(n-1)}{2!}, \qquad {}^nC_3 = \dfrac{n\,(n-1)\,(n-2)}{3!}$

However, we can also see that

$(1 + x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n = \sum_{r=0}^{n} {}^nC_rx^r$

Illustration 2:

Expand (i) $(1 - 2x)^4$ (ii) $(1 + x + x^2)^3$ in powers of x

(i) $(1 - 2x)^4 = (1 + (-2x))^4$

$= 1 + {}^4C_1\,(-2x)^1 + {}^4C_2\,(-2x)^2 + {}^4C_3\,(-2x)^3 + {}^4C_4\,(-2x)^4$

$= 1 + 4\,(-2x) + 6(4x^2) + 4(-8x^3) + 16x^4$

$= 1 - 8x + 24x^2 - 32x^3 + 16x^4$

(ii) $(1 + x + x^2)^3 = [1 + (x + x^2)]^3$

$= 1 + 3(x + x^2) + 3(x + x^2)^2 + (x + x^2)^3$

$= 1 + 3x + 3x^2 + 3(x^2 + 2x^3 + x^4) + (x^3 + 3x^4 + 3x^5 + x^6)$

$= 1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6$

Illustration 3: Find the coefficient of x^8 in $(x^2 + {}^{2y}/_x)^{10}$

Solution: The (r + 1)th term in the expansion of $(x^2 + {}^{2y}/_x)^{10}$ is

${}^{10}C_r\,(x^2)^{10-r}\,({}^{2y}/_x)^r$

${}^{10}C_r\,2^r\,y^r\,x^{20-2r-r}$

Thus, for the term in x^8, we have

${}^{10}C_r\,2^r\,y^r\,x^{20-3r}$, $20 - 3r = 8$, $12 = 3r \Rightarrow r = 4$

$\therefore$ The required coefficients is

${}^{10}C_4\,2^4\,y^4 = \dfrac{10.9.8.7!}{4!\,6!}\,16y^4 = 210.16y^4 = 3360y^4$

$\therefore$ The coefficient of x^8 in $(x^2 + {}^{2y}/_x)^{10} = 3360y^4$

BINOMIAL EXPANSION AND APPLICATIONS

Illustration 4: Obtain the first five terms in the expansion of $(1 + x)^{½.}$ Hence, evaluate $\sqrt{1.03}$ to 5 significant figures.

Solution: For this case n = ½ and so by using the binomial expansion

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots$$

$$(1 + x)^{½} = 1 + ½x + \frac{½(½ - 1)}{2}x^2 + \frac{½(½ - 1)(½ - 2)}{2.3}x^3 + \frac{½(½ - 1)(½ - 3)}{1.2.3.4}x^4 + \ldots$$

$$= \quad 1 + ½x + ⅛x^2 + {}^1/_{16}x^3 - {}^5/_{128}x^4 + \ldots$$

To evaluate $\sqrt{1.03}$, we have

$$\sqrt{1.03} = (1.03)^{½} = (1+0.03)^{½} = 1 + ½(0.03) - ⅛(0.03)^2 + {}^1/_{16}(0.03)^3 - {}^5/_{128}(0.03)^4$$

$$= 1 + 0.015 - 0.0001125 + 0.000001687 - 0.000000031$$

$$= 1.0148892$$

$$\approx 1.0149 \text{ to 5 significant figures.}$$

Illustration 5: Find the values of a, if the coefficient of x^2 in the expansion of $(1 + ax)^4 (2 - x)^3$ is 6.

Solution: $(1 + ax)^4 = 1 + 4(ax) + 6(ax)^2 + 4(ax)^3 + (ax)^4$

$$(2 - x)^3 = [2 + (-x)]^3 \quad = 2^3(-x)^0 + 3.2^2(-x)^1 + 3(2)(-x)^2 + (-x)^3$$

$$= 8 - 12x + 6x^2 - x^3$$

$$(8 - 12x + 6x^2 - x^3)(1 + 4ax + 6a^2x^2 + 4a^3x^3 + a^4x^4) = (1 + ax)^4(2 - x)^3$$

The coefficient of x^2 is:

$$48a^2x^2 - 48ax^2 + 6x^2 \Rightarrow \quad 6 - 48a + 48a^2 = 6$$

$$8a^2 - 8a = 0$$

$$8a(a - 1) = 0$$

$$\Rightarrow a = 0 \text{ or } a = 1$$

Illustration 6: Find the term independent of x in the expansion of $(x - \frac{1}{2}x)^{12}$

Solution:

The general term will be ${}^{12}C_r\, x^{12-r}\, (-\frac{1}{2}x)^r = {}^{12}C_r\, x^{12-r}\, ({}^{1}/_{x}r)\, (-\frac{1}{2})^r$

Now $x^{12-r}\, ({}^{1}/_{x}r) = x^{12-2r}$

If the term is to be independent of x, then $12 - 2r = 0$ or $r = 6$ and this term (the 7th term) is then

$$ {}^{12}C_6\, x^6\, (-\tfrac{1}{2}x)^6 = {}^{12}C_6\, x^6\, ({}^{1}/_{x}6)\, (-\tfrac{1}{2})^6 $$

$$ = \frac{12!}{6!6!} \times \frac{1}{64} = \frac{12.11.10.9.8.7}{1.2.3.4.5.6} \times \frac{1}{64} = \frac{231}{16} $$

$\therefore$ The term independent of x (or the constant term) is ${}^{231}/_{16}$.

Illustration 7: If x is so small that its fourth and higher powers may be neglected, show that $(1 + x)^{1/4} + (1 - x)^{1/4} = a - bx^2$, and find the numbers a and b.

Hence, by putting $x = {}^{1}/_{16}$, show that $(17^{1/4} + 15^{1/4}) \approx 3.9985$

Solution:

$(1 + x)^{1/4} = 1 + \frac{1}{4}x - {}^{3}/_{32}\, x^2 + {}^{7}/_{128}\, x^3 \ldots$

$(1 - x)^{1/4} = 1 - \frac{1}{4}x - {}^{3}/_{32}\, x^2 - {}^{7}/_{128}\, x^3 \ldots$

$(1 + x)^{1/4} + (1 - x)^{1/4} = 1 + \frac{1}{4}x - {}^{3}/_{32}\, x^2 + {}^{7}/_{128}\, x^3 \ldots + 1 - \frac{1}{4}x - {}^{3}/_{32}\, x^2 - {}^{7}/_{128}\, x^3 \ldots$

$= 2 - {}^{3}/_{16}x^2 = a - bx^2$

$\Rightarrow a = 2,\ b = {}^{3}/_{16}$

Putting $x = {}^{1}/_{16}$

$$ (1 + x)^{1/4} + (1 - x)^{1/4} = (1 + {}^{1}/_{16})^{1/4} + (1 - {}^{1}/_{16})^{1/4} $$

$$ = \left(\frac{17}{16}\right)^{1/4} + \left(\frac{15}{16}\right)^{1/4} = \frac{17^{1/4}}{16^{1/4}} + \frac{15^{1/4}}{16^{1/4}} = \frac{17^{1/4}}{2} + \frac{15^{1/4}}{2} $$

$$ = \tfrac{1}{2}(17^{1/4} + 15^{1/4}) $$

$$ (1 + x)^{1/4} + (1 - x)^{1/4} = \tfrac{1}{2}(17^{1/4} + 15^{1/4}) = 2 - {}^{3}/_{16}x^2 $$

If $x = {}^{1}/_{16}$

$= \frac{1}{2}(17^{1/4} + 15^{1/4}) = 2 - {}^{3}/_{16}\, ({}^{1}/_{16})^2$

$= \frac{1}{2}(17^{1/4} + 15^{1/4}) = \dfrac{8189}{4096}$

$17^{1/4} + 15^{1/4} = \frac{8189}{4096} \times 2 = 3.998535156$

$\therefore\ 17^{1/4} + 15^{1/4} \approx 3.9935$

Illustration 8: How many arrangements of letters can be made by using all the letters the word **ALGEBRA?** In how many ways of these arrangement will the a's be separated by at least one other letter?

Solution: Here we have 7 letters of which two are similar. The required number of arrangements is:

$$r = \frac{n!}{p!} = \frac{7!}{2!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 2520$$

If we treat the two a's as one letter, the number of arrangements in which a's are together is (6)! $= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

The number of arrangements in which the a's are separated is the difference, $2520 - 720 = 1800$

Illustration 9: In how many ways can four boys be chosen from six?

Solution: The number of permutation of 4 boys from 6 is ${}^6P_4 = 6 \times 5 \times 4 \times 3$

Hence, the required number of selections is

$${}^6C_4 = \frac{6P4}{4!} = \frac{6 \times 5 \times 4 \times 3}{4!} = \frac{6 \times 5 \times 4 \times 3}{4 \times 3 \times 2 \times 1} = 15$$

Exercises**:**

(1) (a) Evaluate, ${}^{18}C_{16}$, ${}^{11}C_9$, ${}^8C_4$, ${}^8P_4$, ${}^7P_5$ (b) Expand (i) $(a - b)^6$ (ii) $(3x + 2)^5$

(2) In the expansion of $(ax - bx^{-2})$ 8, the coefficients of x^2 and x^{-1} are the same. Show that $a + 2b = 0$

(3) (a) Find the first five terms of the expansion of $(1 - x)^{10}$ in ascending powers of x

(b) Show that if x is so small that x^3 and higher powers of x may be neglected $\frac{(1 + 2x)^{3/2} - 4(1 + x)^{½}}{1 + x^2} = -3 + x + 5x2$

(4) (a) Simplify the following (i) $\frac{19!}{7!4!} + \frac{19!}{8!13!}$ (ii) $\frac{16!}{9!7!} + \frac{2.16!}{10!6!} + \frac{16!}{11!5!}$

(b) Show that $r! \times {}^nC_r = {}^nP_r$, Hence evaluate $\underline{{}^{52}C_{13}}$

$^{52}P_{13}$

(5) (a) Find the term in x^2 and the term independent of x in the expansion of $\left(x - \frac{1}{2x}\right)^{12}$

(b) Find the values of n for which the coefficients of x^4, x^5 and x^6 in the expansion of $(1 + x)^n$ are in arithmetical progression.

(6) (a) How many numbers of four digits can be formed from the digits 1, 2, 3, 4 when each digit can be repeated four times.

(b) Express $\frac{7 + x}{1 + x + x^2 + x^3}$ in partial fraction and obtain an expression of the function in the form: $a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$

Find the values of the coefficients as far as a_5. Show that

$^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

(7) (a) A sub-committee of five including a chairman is to be chosen from a main committee of ten members. If the chairman is to be a specified member of the main committee, in how many ways can this be done?

(b) Employ the binomial theorem to evaluate $(0.998)^{40}$ correct to four places of decimals.

(c) In the binomial expansion of $(1 + x)^n$, where n is a positive integer, the coefficient of x^4 is 1½ times the sum of the coefficients of x^2 and x^3. Find the value of n and determine these three coefficients.

(8) (a) Show that if x is so small that x^3 and higher power of x can be neglected $\sqrt{\left(\frac{1 + x}{1 - x}\right)} = 1 + x + ½ x2$ by putting $x = {}^1/_7$. Show that $\sqrt{3} = \frac{196}{113}$

(b) (i) Obtain the first four terms in the expansion of $(1 + x)^{1/3}$. Hence find the cube root of 1.012 correct to seven decimal places.

(ii) If the first three terms in the expansion of $(1 + x)^p (1 - x)^q$, where p and q are positive integers are $1 + 3x - 6x^2$, find the values of p and q.

(9)

(a) (i) Evaluate $\frac{12!}{10! \times 2!}$ (ii) show that 7C3 + 7C2 = 8C3 and generalize this result by completing the general result nCr = …….

(ii) Find the number of ways in which the letters of the word **SHALLOW** can be arranged, if the two **L's** must not come together.

(c) The ratio of the third to the fourth term in the expansion of $(2 + 3x)^n$ in ascending power of x is 5:14 when $x = {}^2/_5$. Find n.

(d) Calculate a if the coefficient of x^3 in $(a + 2x)^5$ is 320.

MTS 105 LECTURE 5: SEQUENCE AND SERIES

1.0 SEQUENCE

A sequence is an endless succession of numbers placed in a certain order so that there is a first number, a second and so on. Consider, for example, the arrangement

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\dots\dots \frac{1}{n}, \dots..$$

This is a sequence where n^{th} number is attained by taking the reciprocal of n. We can have many more examples, such as

$$1,2,3, \dots\dots\dots\dots, n, \dots.$$

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\dots\dots \frac{n}{n+1}, \dots.$$

$$-1, 1, -1, 1, \dots..(-1)^2 \dots\dots\dots \text{ Etc}$$

So, in general, a sequence is of the form

$$V_1, V_2, V_3, \dots\dots\dots\dots\dots V_n, \dots..$$

Where V_i's are real numbers and each V_i has a duplicate position. This sequence in the notational form is written as $<V_n>$ or $\{V_n\}$ where V_n is the n^{th} (general) term of the sequence.

Example 1:

1. Given that $U_r = 2_r + 1$, then

$V_1 = 2(1) + 1 = 3$

$V_2 = 2(2) + 1 = 5$

$V_3 = 2(3) + 1 = 7$

- - - - -

- - - - -

Example 2:

2. Given the sequence $1, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\dots\dots$, find the N^{th} term

Solution: By , we see that the sequence of terms can be written as follows:

$V_1 = 1 = \frac{1}{1^2},$

$V_3 = \frac{1}{9} = \frac{1}{3^2},$

Hence the n^{th} term $V_n = \frac{1}{n^2},$

$V_3 = \frac{1}{9} = \frac{1}{1^2},$

$V_r = \frac{1}{r^2},$

$\therefore V_r = \frac{1}{r^2},$

3. Given the sequence 1,4,9,16,25, ……………….. Find the V_r

$V_1 = 1 = 1^2$

$V_2 = 4 = 2^2$

$V_3 = 9 = 3^2$

$V_4 = 16 = 4^2$

$V_5 = 25 = 5^2$

$\therefore V_r = r^2$

4. Find the first five terms of the sequence defined as follows

$V_1 = 1$, $V_2=3$, $V_r=3V_{r-1} - V_{r-2}$

$V_3 = 3V_2 - V_1 = 3(3) - 1 = 8$

$V_4 = 3V_3 - V_2 = 3(8) - 3 = 21$

$V_5 = 3V_4 - V_3 = 3(21) - 8 = 55$

$\therefore$ The first five terms are 1,3,6,21,55

1.1 SERIES

A Series is obtained by forming the sum of the terms of a sequence. A finite series is obtained if a first number of terms of the sequence are summed, otherwise is an infinite series.

The sum of the first n^{th} terms of the sequence V_1, V_2, …..V_n is generally denoted by S_n

$$S_n = V_1 + V_2 + \ldots\ldots\ldots\ldots + V_n = = \sum_{r=1}^{n} V_r$$

Here we will be considering basically the following series:

1.1.1 Arithmetic Sequence (or Arithmetic Progression)

An Arithmetic progression (A.P) is a sequence of terms that increases by a constant amount which may be positive or negative. This constant amount is known as the common difference of the series and is usually denoted by d.

The following series are all Arithmetic Progression.

1,2,3,4, ………	Common difference	= +1
$\frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2,$………	Common difference	$= +\frac{1}{4}$
6,3,0,-3 ………..	" "	= - 3
$-2, -1\frac{1}{2}, -1, -\frac{1}{2},$ …	" "	$= +\frac{1}{2}$

To ascertain if a given sequence be an A.P, it is necessary to subtract from each term (except the first) the preceeding term. If the results in all cases be the same, the given series will be an A.P, where common difference is this common result.

N^{th} Term of an A.P

First term = a = a + (1 – 1)d

Second term = a + d = a + (2 – 1)d

Third term = a + 2d = a + (3 – 1)d

From the result, it is obviouse that the n^{th} term is a + (n – 1)d.

Arithmetic Mean

To find the arithmetic mean between a and b. Let x be the required A.M. Then a, x, b will be three consecutive terms of an A.P. The common difference of the A.P is (x – a) and also (b – x).

∴ $x - a = b - x$

i.e $2x = a + b$

$$x = \frac{a + b}{2}$$

More generally, the n arithmetic means between a and b are the n quantities that, when inserted between a and b, form with them (n + 2) successive times of an A.P.

N Arithmetic Means between a and b

Let d be the common difference of the A.P formed. Then b is the $(n + 2)^{th}$ term of the A.P.

∴ $b = a + (n + 1)\, d$

∴ $d(n + 1)\ b - a$

∴ $d = \frac{b - a}{(n + 1)}$

The required arithmetic means are a + d, a + 2d, ..., a + nd,

Where $d = \frac{b - a}{(n + 1)}$

The Sum S of n terms of an A.P

To find the sum S of n terms of an A.P whose first term is a and whose common difference is d, we must note that the last term of the A.P will be a + (n – 1)d.

∴ $S = a + (a + d) + (a + 2d) + \ldots\ldots\ldots\ldots + [a + (n - 1)d]$ …………………. 1

Reversing the series, we get

$S = [a + (n - 1)d] + [a + (n - 2)d] + [a + (n - 3)d] + \ldots\ldots\ldots + a$ ………….2

Hence, (1) + (2) gives:

$2S = [2a + (n - 1)d] + [2a + (n - 1)d] + [2a + (n - 1)d] + \ldots\ldots$ to nterms

$= n\ [2a + (n - 1)d[$

∴ $S = \frac{n}{2}\{2a + (n - 1)d\}$

Note: Using l for the last term, {a + (n – 1)d}, the result can be written as

$S = \frac{n}{2}(a + l)$

Examples

1. The sum of three consecutive terms of an A.P is 18, and their products is 120. Find the term

Solution:

Let the term be: a – d, a, a + d (we should take the term a, a+d, a+2d but it is simplier to take a – d, a, a + d)

∴ $a - d + a + a + d = 18$

$3a = 18$

∴ $a = 6$

The Product = $(a - d)\ a\ (a +) = 6(6 - d)(6 + d) = 120$

∴ $6(6^2 - d^2) = 120$

∴ $6(36 - d^2) = 120$

$36 - d^2 = 20$

$\therefore \quad d^2 = 36 - 20$

$\therefore \quad d = \pm 4$

If d = 4, the terms are 2, 6, 10 and if d = - 4 they are 10, 6, 2. The numbers are the same but form two different A.Ps.

2. The first term of an arithmetic series is 7, the last term is 70, and the sum is 385. Find the number of terms in the series and the common difference.

Solution

Here a = 7, l = 70, Sn = 385

But $Sn = \frac{n}{2}(a + l)$

$\therefore \quad 385 = \frac{n}{2}(7 + 70) = \frac{77n}{2}$

$\therefore \quad 77n = 770$

$n = 10$

Since $l = 70$, then $7 + (10 - 1)d = 70$

$\therefore \quad 7 + 9d = 70$

$9d = 63 \Rightarrow d = 7$

$\therefore$ n = 10 is member of terms of the series and the common difference = 7

3. Find the number of terms in an A.P whose first term is 5, common difference 3, and sum 55.

Solution:

Let n be the required number of terms

$\therefore \quad 55 = \frac{1}{2}n(7 + 3n)$

$\therefore \quad 110 = 7n + 3n^2$ i.e $3n^2 + 7n - 110 = 0.$

$\therefore \quad (3n + 22)(n - 5) = 0, \therefore n = \frac{-22}{3}$ or 5

But n must be a positive integer $\therefore$ n = 5

4. The sum of the first n terms of a series is $2n^2 - n$. Find the nth term and show that the series is an A.P.

Solution:

Using $n = 1$ in the sum for n terms, it is seen that the first term $= 2 – 1 = 1$.

Using $n = 2$, the sum of the first two terms $= 8 – 2 = 6$, therefore the second term is 5

Using $n = 3$, the sum of the first three terms $= 18 – 3 = 15$, therefore the third term

$= 15 – 6 = 9$

Replacing n by $(n – 1)$ the sum of the first $(n – 1)$ terms is

$2 (n -1)^2 – (n – 1) = 2n^2 – 4n + 2 – n + 1$

$= 2n^2 – 5n + 3$

$\therefore$ n^{th} term = sum of first nterms – sum of first $(n – 1)$ term

$= 2n^2 – n – (2n^2 – 5n + 3)$

$= 2n^2 – n – 2n^2 + 5n – 3$

$= 4n – 3$

The series is 1,5,9,…… $(4n – 3)$, which is an A.P of common difference 4.

5. The sum of five numbers in A.P is 25 and the sum of the series be d, then the terms are $(a – 2d)$, $(a – d)$, a, $a + d$, $a + 2d$.

From the question

$(a – 2d) + (a – d) + a + (a + d) + (a + 2d) = 25$

$\therefore$ $5a = 25$

$\therefore$ $a = 5$

Also, $(a – 2d)^2 + (a – d)^2 + a^2 + (a + d)^2 + (a + 2d)^2 = 165$

$a^2 – 4ad + 4d^2 + a^2 – 2ad + d^2 + a^2 + (a^2+2ad+d^2)+ (a^2+4ad+4d^2) = 165$

$\therefore$ $5a^2 + 10d^2 = 165$

i.e $a^2 + 2d^2 = 33$

$2d^2 = 8$

$\therefore$ $d^2 = 4$

$\therefore$ $d = \pm 2$

Hence, the series is 1,3,5,7,9 ….

1.2.2 Geometric Sequence (Geometric Progression)

The geometrical progression (G.P) is a sequence of terms that increase or decrease in a constant ratio. This constant ratio is known as the common ratio of the series and is usually denoted by r.

The following examples are all geometrical progressions (G.P)

(i) 2, 4, 8, 16, ………………….. Common ratio $+ 2$

(ii) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16},$ …………………… Common ratio $+ \frac{1}{2}$

(iii) $-\frac{1}{3}, 1, -3, 9,$ ……………….. Common ratio $- 3$

(iv) $- 2, \frac{1}{2}, -\frac{1}{3}, \frac{1}{32},$……………….. Common ratio $-\frac{1}{4}$

If, in a given sequence, the ratio of each term to the preceding term is the same for all terms, the sequence must be a G.P with this ratio as the common ratio.

nth term of a G.P

First term = $a = a.r^0 = ar^{1-1}$

Second term = $ar = a.r^{2-1}$

Third term = $ar^2 = ar^{3-1}$

From these it can be seen that the nth term is $a.r^{n-1}$.

The geometric mean (G.M) between two quantities a and b is that quantity which when inserted between a and b, forms with them three successive terms of a G.P.

Geometric Mean Between a and b

Let x be the required G.M. Then, a,x,b will be three successive terms of a G.P. The common ratio of the G.P will be x/9 and b/x

∴ $\frac{x}{9} = \frac{b}{x}$

∴ $x^2 = ab$

∴ $x = \pm \sqrt{ab}$

More generally, the n geometric means between a and b are the n quantities that when inserted between a and b form with them (n + 2) terms (successive) of a G.P.

n Geometric means between a and b

Let r be the common ratio of the G.P formed. Then, b is the (n + 2)th term of the G.P.

∴ $b = ar^{n+1}$

i.e $r^{n+1} = b/a$

∴ $r = \sqrt[n+1]{b/a}$

Using this value of r, the required geometric means will be $ar, ar^2,$ …………, ar^2

Sum of the first n terms of a G.P

To find the sum of the first n terms of a G.P, whose common ratio is r and first term a.

Let S_n be the required sum. Then,

$$Sn = a + ar + ar^2 + \ldots\ldots\ldots + ar^{n-1} \ldots\ldots \qquad (1)$$

$$\therefore \quad rSn = ar + ar^2 + ar^3 + \ldots\ldots + ar^{n-1} + ar^n \ldots\ldots \qquad (2)$$

(1) – (2) gives, $Sn(1 - r) = a - ar^n$

$$= a(1 - r^n)$$

$$\therefore \quad Sn = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$$

Note:

$$Sn = \begin{cases} \frac{a(1-r^n)}{1-r}, & /r/ <1 \\ \frac{a(r^n-1)}{r-1}, & /r/ >1 \end{cases}$$

The value that Sn approaches as $n \longrightarrow \infty$ is known as its sum to infinity ($S\infty$)

From the previous result

$$Sn = a\,\frac{(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

Therefore as $n \longrightarrow \infty$ (i.e as n approaches infinity) $r^n \longrightarrow 0$, if /r/ <1, and the second term becomes negligible, whilst r^n n $\longrightarrow \pm\infty$ when /r/ >1 and Sn is then infinite.

Thus, when /r/ < 1, $S\infty = \frac{a}{1-r}$

If the sum to infinity is finite, as in this case, the series is said to be convergent.

When /r/ > 1, $S\infty = \pm\infty$ and the series is divergent

If r = 1, the series is a + a + a + ….. and so Sn = na

Hence, $Sn \longrightarrow +\infty$ if a > 0 but $Sn \longrightarrow -\infty$ if a < 0 (both dgt)

If r = - 1, the series is a, - a + a – a + ….. Hence Sn = 0 or a

Example

1. Insert three geometric means between $2\frac{1}{4}$ and $\frac{4}{9}$

Solution

Let r be the common ratio of the G.P formed. Since $\frac{4}{9}$ is the fifth term of the G.P

$$\frac{4}{9} = \frac{9}{4}r^4 \qquad \therefore \quad r^4 = \frac{16}{81}$$

$$\therefore \quad r^2 = \frac{4}{9} \text{ (r real)} \qquad \therefore \quad r = \pm\frac{2}{3}$$

Therefore required geometric means are $^3/_2$, 1, $^2/_3$ or $-^3/_2$, 1, $-^2/_3$

2. In a geometric progression, the first term is 7, the last term 448 and the sum 889. Find the common ratio.

Solution:

Let r be the common ratio, n the number of terms, and Sn the sum of nterms.

$\therefore$ Sn = 889 $= \frac{7(1-r^n)}{1-r}$ .... **(1)**

Also $448 = 7r^{n-1}$ .... **(2)**

Using (2) in (1), $889 = \frac{7-7r^n}{1-r} = \frac{7-448r}{1-r}$

$889(1 - r) = 7 - 448r$

$\therefore$ $889 - 889r = 7 - 448r$

$\therefore$ $882 = 441r$

$\Longrightarrow$ $r = 2$

3. Find the sum of the first six terms of the geometric series whose thrid term is 27 and whose sixth term is 8.

Find how many terms of this series must be taken if their sum is to be within $^1/_{10}$% of the sum to infinity.

Solution:

Let r be the common ratio of the series and a the first term.

$\therefore$ $ar^2 = 27$ .... (1)

$ar^5 = 8$ .... (2)

(2) ÷ (1) gives $r^3 = {}^8/_{27}$ $\therefore$ $r = \frac{2}{3}$

Using this in (1), $9\,x\,\frac{4}{9} = 27$ $\therefore$ $a = \frac{243}{4}$

The sum of the first six terms of this series

$$= \frac{9(1-r^n)}{1-r} = \frac{243}{4}\,\frac{\{1-\left(\frac{2}{3}\right)^6\}}{1-\frac{2}{3}} = \frac{243}{4}\,\frac{\{1-\frac{26}{36}\}}{\frac{1}{3}}$$

$$= \frac{729}{4}\left\{1-\frac{26}{36}\right\} = \frac{729}{4} - \frac{729}{4}\,x\,\frac{26}{36}$$

$$= \frac{729}{4} - 16 = \frac{665}{4}$$

Let n be the number of terms required so that their sum shall be $^1/_{10}$% less than the sum to infinity. $S\infty$.

Now $S\infty = \frac{9}{1-r}$, *and* Sn $= \frac{a(1-r^n)}{1-r} = \frac{9}{1-r} - \frac{ar^n}{1-r}$

$\therefore \quad S\infty - Sn = \frac{ar^n}{1-r}$

But $S\infty$ - Sn = $\frac{1}{10}$ percent of $S\infty$

$\therefore \quad \frac{S\infty - Sn}{S\infty} = \frac{1}{1000}$

$\therefore \quad \frac{ar^n}{1-r} \div \frac{a}{1-r} = \frac{1}{1000}$ i.e. $r^n = \frac{1}{1000}$

(i.e.) $\left(\frac{2}{3}\right)^n = \frac{1}{1000}$

Taking logs to the base of 10, we get

n[*log*2 – *log*3] = - 3

$\therefore \quad n = \frac{3}{log_{10} 3 - log_{10} 2} = \frac{3}{0.47712 - 0.30103} = \frac{3}{0.17609} = 17.03$

$\Longrightarrow$ Regd no is 18 (i.e. next +ve integer > 17)

The powers of the First n Natural Numbers

Note: The method adopted in the summation of the second and third powers of the first n natural numbers is the method of undetermined coefficients, and consisted of equating identically the given series to $A + B_n + C_n^2 + \ldots\ldots\ldots$ and then determining the values of the unknown constants A, B, C etc. by the properties of identities.

a. The first n natural numbers form an A.P whose sum S_1 has been shown to be n(n+1)/2.

b. Let the sum of the squares of the first n natural numbers be S_2 and

$$1^2 + 2^2 + 3^2 + \ldots + n^2 \equiv A + B_n + C_n^2 + D_n^3 + \ldots\ldots\ldots\ldots \qquad (1)$$

Replacing n by (n + 1) in this identity

$$1^2 + 2^2 + 3^2 + \ldots + (n+1)^2 \equiv A + B((n+1) + C(n+1)^2 + D(n+1)^3 + \ldots \qquad (2)$$

(2) – (1) gives:

$$(n+1)^2 \equiv B + C(2n+1) + D(3n^2+3n+1) + \ldots\ldots\ldots\ldots\ldots\ldots \qquad (3)$$

In the identity (3) the singular power of n on the left-hand side (L.H.S) is the second, and therefore the highest power of n on the right-hand side (R.H.S) must be the second and all other coefficients after D must varies.

Equating coefficient of n in (3)

n^2 : $1 = 3D$ $\quad \therefore \; D = \frac{1}{3}$

n : $2 = 3D + 2C = 1 + 2C$ $\quad \therefore \; C = \frac{1}{2}$

Unity: $1 = B + C + D = B + \frac{1}{2} + \frac{1}{3} \quad \therefore \quad B = \frac{1}{6}$

$\therefore \quad 1^2 + 2^2 + 3^2 + \ldots + n^2 \equiv A + \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3$

Using $n = 1$, $\quad 1^2 = A + \frac{1}{6} + \frac{1}{2} + \frac{1}{3}$

$\therefore \quad A = 0$

Thus, $\quad 1^2 + 2^2 + 3^2 + \ldots + n^2 = + \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3$

$$= \frac{n}{6}(1 + 3n + 2n^2)$$

$$= \frac{n(n+1)(2n+1)}{6}$$

Alternative Method

It is known that, for all values of n,

$(n + 1)^3 - n^3 \equiv 3n^2 + 3n + 1$ ……………………… (1)

Replacing n by (n-1), (n-2), ……….., 2, 1 in succession,

$n^3 - (n - 1)^3 \equiv 3(n - 1)^2 + 3(n - 1) + 1$ ………… (2)

$(n - 1)^3 - (n - 2)^3 \equiv 3(n - 2)^2 + 3(n - 2) + 1$ ……. (3)

- - - - - - - - - -

$3^3 - 2^3 \equiv 3.2^3 + 3.2 + 1$ ………………………… (n-1)

$2^3 - 1^3 \equiv 3.1^3 + 3.1 + 1$ ………………………… (n)

Adding these n identities

$$(n + 1)^3 - 1^3 \equiv 3(1^2 + 2^2 + \ldots + n^2) + 3(1 + 2 + 3 + \ldots + n) + 2$$

$$\equiv 3S_2 + \frac{3}{2}n(n+1) + n$$

$\therefore \quad 3S_2 \equiv (n + 1)^3 - 1^3 - 3\frac{n}{2}(n + 1) - n$

$= n^3 + 3n^2 + 3n - 3\frac{n}{2}(n + 1) - n$

$= \frac{n}{2}\{2n^2 + 6n + 6 - 3n - 3 - 2\}$

$= \frac{n}{2}\{2n^2 + 3n + 1\} = \frac{n}{2}(n + 1)(2n + 1)$,

$\therefore \quad S_2 = \frac{n(n+1)(2n+1)}{6}$

(c) Let S_3 be the sum of the cubes of the first n natural numbers and

$$1^3 + 2^3 + 3^3 + \ \ldots + n^3 \equiv A + Bn + Cn^2 + Dn^3 + En^4 \ldots\ldots \qquad (1)$$

Replacing n by (n+1) in this identity,

$$1^3 + 2^3 + 3^3 + \ldots + n^3 + (n+1) \equiv A + B(n+1) + C(n+1)^2 + D(n+1)^3 + E(n+1)^4 + \ldots \qquad (2)$$

(2) – (1) gives

$$(n+1)^3 \equiv B + C(2n+1) + D(3n^2 + 3n + 1) + E(4n^3 + 6n^2 + 4n + 1) + \ldots\ldots\ldots \qquad (3)$$

The highest power of n on the L.H.S of (3) is the third, and therefore all coefficients after E on the R.H.S of (3) must vanish.

Equating coefficients in identity (3),

n^3 : $1 = 4E$ $\therefore$ $E = {}^{1}/_{4}$

n^2 : $3 = 6E + 3D = \frac{3}{2} + 3D$ $\therefore$ $3D = \frac{3}{2}$ $\therefore$ $D = \frac{1}{2}$

n : $3 = 4E + 3D + 2C = 1 = 1 + \frac{3}{2} + 2C$ $\therefore$ $\frac{1}{2} = 2C,$ $\therefore$ $C = \frac{1}{4}$

Unity: $1 = B + C + D + E = B + \frac{1}{4} + \frac{1}{2} + \frac{1}{4},$ $\therefore$ $B = 0$

Hence, $1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 \equiv A + \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4.$

Using n =1 in this,

$1^3 = A + \frac{1}{4} + \frac{1}{2} + \frac{1}{4},$ $\therefore$ $A = 0,$

$$\therefore \quad S_3 = \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4 \equiv \frac{n^2}{4}\left(1 + 2n + n^2\right) \equiv \left\{\frac{n}{2}(n+1)\right\}^2 = S_1^2$$

As in (b), the alternative method consists of using

$$(n+1)^4 - n^4 \equiv 4n^3 + 6n + 4n + 1$$

and replacing n by (n – 1), (n – 2), ……………, etc, in succession and then introducing the values of S_1 and S_2 already determined.

Example: Find the sum of the series

(i) $1^2 - 2^2 + 3^2 - 4^2 + \ldots\ldots\ldots\ldots\ldots$ to 2n terms

(ii) $1.3^2 + 2.4^2 + 3.5^2 + \ldots\ldots\ldots\ldots\ldots$ to n terms

(iii) $1^3 + 3^3 + 5^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ to n terms

Solution

(i) Let the required sum the S_2n

$$\therefore \quad S_2n = (1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + \text{ to } 2n \text{ terms}) - 2(2^2 + 4^2 + 6^2 + \ldots\ldots\ldots + \text{ to } n\text{terms})$$

= $\sum_{r=1}^{r=2n} r^2 - 8(1^2 + 2^2 + 3^2 + \cdots + to\ n\ term)$

= $\sum_{r=1}^{2n} r^2 - 8\sum_{r=1}^{n} r^2$

= $\frac{2n(2n+1)(4n+!)}{6} - \frac{8n(n+1)(2n+1)}{6}$ {Using formular fo S_2}

= $\frac{n(2n+1)}{3}\{4(n+1) - 4(n+1)\} = \frac{-(2n+1).3}{3}$

= - n(2n + 1)

(ii) The r^{th} term of the series = $r(r+2)^2 = r^3 + 4r^2 + 4r$, therefore sum of nterms of the series

= $\sum_{r=1}^{n}(r^3 + 4r^2 + 4r)$

= $\sum_{r=1}^{n} r^3 + 4\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r$

= $\left\{\frac{n(n+1)}{2}\right\}^3 - \frac{4n(n+1)(2n+1)}{6} + 4 \times \frac{n}{2}(n+1)$

= $\frac{n}{12}(n+1)\{3n(n+1) + 8(2n+1) + 24\}$

= $\frac{n(n+1)}{12}\{3n^2 + 19n + 32\}$.

(iii) The r^{th} term of the series is $(2r-1)^3 \equiv 8r^3 - 12r^2 + 6r - 1$, therefore sum of n terms

= $\sum_{r=1}^{n}(8r^3 + 12r^2 + 6r - 1)$

= $8\sum_{r=1}^{n} r^3 - 12\sum_{r=1}^{n} r^2 + 6\sum_{r=1}^{n} r - n)$

= $8\left\{\frac{n(n+1)}{2}\right\}^2 - 12\frac{n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{2} - n$

= $2n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n$

= $n\{(2n^3+4n^2+2n) - (4n^2+6n+2) + (3n+3) - 1\}$

= $n\{2n^3 - n\} = n^2(2n-1)$.

Note: Example (i) can be solved readily as follows:

$1^2 - 2^2 + 3^2 - 4^2 + \ldots\ldots\ldots\ldots\ldots\ldots + (2n-1)^2 - (2n)^2$

= $(1^2 - 2^2) + (3^2 - 4^2) + \ldots\ldots\ldots\ldots\ldots + \{(2n-1)^2 - (2n)^2\}$

= $-1.3 - 1.7 - 1.11 - \ldots\ldots\ldots\ldots\ldots\ldots - 1(4n-1)$

= $-\frac{1}{2}n(4n+2) = -n(2n+1)$

1. Factor and Remainder Theorems

Factor Theorem: *(x – a)* is a factor of a polynomial *f(x)* if *f(a) = 0*.

Remainder Theorem: The remainder when a polynomial *f(x)* is divided by *(x – a)* is *f(a)*.

- (ax + b) is a factor of a polynomial f(x) if $f\left(\frac{-b}{a}\right) = 0$.

Examples:

Show that (x – 3) is a factor of $x^3 - 2x^2 - 5x + 6$ and find the other two factors.

Solution:

Let f(x) = $x^3 - 2x^2 - 5x + 6$.

(x – 3) is a factor if f(3) = 0.

f(3) = $3^3 - 2\times 3^2 - 5\times 3 + 6 = 27 - 18 - 15 + 6 = 0$.

So (x – 3) is a factor.

To find the other factors we divide $x^3 - 2x^2 - 5x + 6$ by (x – 3):

$$
\begin{array}{r}
x^2 + x - 2 \\
x-3\,\big)\overline{x^3 - 2x^2 - 5x + 6} \\
\underline{x^3 - 3x^2} \\
x^2 - 5x \\
\underline{x^2 - 3x} \\
-2x + 6 \\
\underline{-2x + 6} \\
0
\end{array}
$$

To find the other factors we have to factorise $x^2+x-2=(x-1)(x+2)$

So x^3-2x^2-5x+6 = (x – 3)(x – 1)(x + 2).

Example 2:

Find the remainder when $2x^3-5x^2+2x+7$ is divided by (x + 2).

Solution:

Let f(x) = $2x^3-5x^2+2x+7$

The remainder when f(x) is divided by (x + 2) is f(-2).

f(-2) = $2(-2)^3+5(-2)^2+2(-2)+7=-16+20-4+7=7$.

Exercises

- When $2x^3-x^2-13x+k$ is divided by x – 2 the remainder is -20. Find k
- Show that (x – 3) is a factor of $x^3+x^2-8x-12$ and find the other two factors.
- Factorise x^3-4x^2+x+6, given that (x – 2) is a factor.

2. PARTIAL FRACTION

It is possible to write the expression $\dfrac{1}{(x+1)(x+2)}$ as partial fractions $\dfrac{1}{(x+1)}-\dfrac{1}{(x+2)}$.

Procedure:

1. Express the denominator in its most simple factorised form, if it is not in this form already. For example, given the expression $\dfrac{1}{x^2+3x+2}$ the first step would be to write it in the form $\dfrac{1}{(x+1)(x+2)}$ and then determine the partial fractions.

2. If the degree of the numerator is equal or greater than that of the denominator then it will be possible to divide the numerator by the denominator, leaving a polynomial expression summed with an expression consisting of a polynomial numerator of lower degree to the polynomial denominator.

For example, given the expression $\frac{2x^2+7x+3}{x^2+3x+2}$ the degree of the numerator is equal to the degree of the denominator.

$$\frac{2x^2+7x+3}{x^2+3x+2}=2+\frac{x+1}{x^2+3x+2}.$$

Examples:

LINEAR FACTORS

Resolve $\frac{1}{(x+1)(x+2)}$ into partial fractions.

Solution:

The expression $\frac{1}{(x+1)(x+2)}$ consists of two linear, unrepeated factors in the denominator.

Let $$\frac{1}{(x+1)(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$$

Where A and B are constants to be determined.

By algebraic addition

$$\frac{1}{(x+1)(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+2)}=\frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

Since the denominators are the same on each side of the identity then the numerators are equal to each other.

$1 = A(x+2) + B(x+1)$

To determine constants A and B, values of x are chosen to make the term in A or B equal to zero.

Let x = -1, then $A = 1$

If x = -2, then $B = -1$

Therefore,

$$\frac{1}{(x+1)(x+2)} \equiv \frac{1}{(x+1)} - \frac{1}{(x+2)}$$

We can also obtain the values of the constants A and B, using the method of EQUATING THE COEFFICIENTS.

$1 = A(x+2) + B(x+1)$

$1 = Ax + 2A + Bx + B = (A+B)x + (2A+B)$

Equating the coefficients in *x;* $0 = A + B$

Equating the coefficients in *constant;* $1 = 2A + B$

Thus, $A = 1, B = -1$

Hence, $\frac{1}{(x+1)(x+2)} \equiv \frac{1}{(x+1)} - \frac{1}{(x+2)}$

REPEATED FACTORS

Given an expression $\frac{2x+3}{(x-2)^2(x+2)}$, we write the partial fractions as

$$\frac{2x+3}{(x-2)^2(x+1)} \equiv \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x+1)}$$

We then follow the process in (1) to obtain the constants A, B and C.

Remark: $\frac{f(x)}{(x+a)^3} \equiv \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3}$

QUADRATIC FACTOR

If the denominator is a combination of a quadratic factor, which does not factorise without introducing imaginary surd terms;

Example,

Resolve $\frac{x^2+5x+3}{(x^2+3)(x+2)}$ into partial fractions

We resolve the expression into partial fractions as

$$\frac{x^2+5x+3}{(x^2+3)(x+2)} \equiv \frac{Ax+B}{(x^2+3)} + \frac{C}{(x+2)}$$

We then follow the process in (1) to obtain the constants A, B and C.

3. INEQUALITIES

An inequality is similar to an equation except that the two expressions have a relationship other than equality, such as $<$, $\leq$, $>$, or $\geq$.

To solve an inequality means to find all values of the variable that make the inequality true.

Rules for Inequalities

1. $A \leq B \Leftrightarrow A + C \leq B + C$

2. A $\leq$ B $\Leftrightarrow$ A - C $\leq$ B – C

3. If C > 0, then A $\leq$ B $\Leftrightarrow$ CA $\leq$ CB

4. If C < 0, then A $\leq$ B $\Leftrightarrow$ CA $\geq$ CB

5. If A $\leq$ B and C $\leq$ D, then A + C $\leq$ B + D

Examples on linear inequality

1. Solve $7x+23<2$

Solution

$$
\begin{aligned}
&7x+23<2\\
&7x+23-23<2-23\\
&7x<-21\\
&x<-3
\end{aligned}
$$

2. Solve $4(2-x)<6-x$

Solution:

$$
\begin{aligned}
&4(2-x)<6-x\\
&8-4x<6-x\\
&8-8-4x+x<6-8-x+x\\
&-3x<-2\\
&\Rightarrow\\
&3x>2\\
&\textit{therefore},\\
&x>\frac{2}{3}
\end{aligned}
$$

Examples on quadratic inequality

1. Solve the inequality $x^2+4x-21<0$.

Solution

$$x^2 + 4x - 21 < 0$$
$$factorise$$
$$(x+7)(x-3) < 0$$

We need to find to find the range of x for $(x+7)(x-3) < 0$ is negative.

So, we consider

$(x+7) < 0$ and $(x-3) > 0$ {Impossible}

$x < -7$ $\quad$ $x > 3$

OR

$(x+7) > 0$ and $(x-3) < 0$

$x > -7$ $\quad$ $x < 3$

$\Rightarrow$

$-7 < x < 3$

Hence the solution is $-7 < x < 3$.

2. Solve the inequality $x^2 + 8x - 33 \geq 0$

Solution

$$x^2 + 8x - 33 \geq 0$$
$$factorise$$
$$(x+11)(x-3) \geq 0$$

We need to find the values of x for which $(x+11)(x-3)$ is positive or zero.

So, we consider

$(x+11) \geq 0$ and $(x-3) \geq 0$

$x \geq -11$ $\quad$ $x \geq -3$

$\Rightarrow$

$x \geq 3$

OR

$(x+11) \le 0$ and $(x-3) \le 0$

$x \le -11$ $\quad$ $x \le -3$

$\Rightarrow$

$x \le -11$

Therefore, the solution is

$$x \le -11 \quad \text{or} \quad x \ge 3$$

Interval form: $(-\infty, -11] \cup [3, \infty)$

3. $x^2 - 10x + 29 \ge 0$

4. $x^2 - 6x \le 0$

5. $x^2 + 2x - 35 \ge 0$

6. $x^2 - 10x - 1 \ge 0$

7. $2x^2 + 12x + 20 < 0$

COURSE CODE: MTS 105

COURSE TITLE: Algebra

NUMBER OF UNITS: 3

COURSE DURATION: Three hours per week

COURSE DETAILS:

Course Coordinator: Dr.I.A Osinuga, Dr. I. O. Abiala

TOPICS: Binomial Theorem,Binomial Series,Binomial Expansion and Applications **Definition**:

A binomial expression is one that contains two terms connected by a plus or minus sign.Thus $(p+q),(a+x)^2,(2x+y)^3$ are examples of binomial expression.

Note:

In order to solve $(a+x)^n$:

1. a decreases in power moving from left to right
2. x increases in power moving from left to right
3. The coefficients of each term of the expansions are symmetrical about the middle coefficient when n is even and symmetrical about the two middle coefficients when n is odd.
4. The coefficients are shown separately below and this arrangement is known as as pascal triangle triangle.A coefficient of a term may be obtained by adding the two adjacent coefficient immediately above in the

previous row. This is shown by the triangle below,where for example, $1 + 3 = 4, 10 + 5 = 15$, and so on.

Pascal triangle method is used for expansion of the form $(a + x)^n$ for integer values of n less than about 8.

n=0: 1

n=1: 1 1

n=2: 1 2 1

n=3: 1 3 3 1

n=4: 1 4 6 4 1

n=5: 1 5 10 10 5 1

n=6: 1 6 15 20 15 6 1

n=7: 1 7 21 35 35 21 7 1

The numbers in the n-th row represent the binomial coefficients in the expansion of $(a + x)^n$.

Example:

Use the pascal's triangle method to determine the expansion of $(a + x)^7$.

Solution

$(a + x)^7 = a^7 + 7a^6x + 12a^5x^2 + 35a^4x^3 + 35a^3a^4 + 21a^2x^5 + 7ax^6 + x^7$

Example:

Determine,using pascal's triangle method,the expansion of $(2p - 3q)^5$.

Solution

$(2p-3q)^5 = (2p)^5+5(2p)^4(-3q)+10(2p)^3(-3q)^2+10(2p)^2(-3q)^3+5(2p)(-3q)^4+$

$(-3q)^5$

$= 32p^5 - 240p^4q + 720p^3q^2 - 1080p^2q^3 + 810pq^4 - 243q^5$

The binomial Series

The binomial series or binomial theorem is a formula for raising binomial expression ton any power without lengthy multiplication.The general binomial expansion of $(a+x)^n$ is given by

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + ... + x^n$$

$$= \sum_{r=0}^{n} \tbinom{n}{r} a^{n-r}x^r, \tbinom{n}{r} = \frac{n!}{r!(n-r)!}$$

where for example,3! denotes $3 \times 2 \times 1$ and is termed factorial 3.

With the binomial theorem,n may be a fraction,a decimal fraction, a positive or a negative integer.

Note:

1. ${}^nC_r = \binom{n}{r} = \frac{n(n-1)(n-2)...(n-r+1)}{r!(n-r)!}$ is called binomial coefficient.

2. Since ${}^nC_r =^n C_{n-r}$,it follows that the coefficients in the binomial expansion are symmetrical about the middle. There is one middle term (i.e the $\frac{n}{2}$-th term) if n is even,and two middle terms (i.e the $\frac{n-1}{2}$-th and $\frac{n+1}{2}$-th term). If n is odd.

3. The term ${}^nC_r a^{n-r}x^r$ is the $(r+1)$-th term.

In the general expansion of $(a+x)^n$,it is noted that the 4th term is

$$\frac{n(n-1)(n-2)}{3!}a^{n-3}x^3$$

The rth term of the expansion is $\frac{n(n-1)(n-2)...n-(r-2)}{(r-1)!}$

If $a=1$ in the binomial expansion of $(a+x)^n$ then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

which is valid for $-1 < x < 1$.

Example:

Use the binomial series to determine the expansion of $(2+x)^7$

Solution:

The binomial expansion is given by

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + ...$$

when $a=2$ and $n=7$

$$(2+x)^7 = 2^7+7(2)^6x+\frac{(7)(6)}{(2)(1)}2^5x^2+\frac{(7)(6)(5)}{3!}2^4x^3+\frac{(7)(6)(5)(4)}{4!}2^3x^4+\frac{(7)(6)(5)(4)(3)}{5!}2^2x^5+\frac{(7)(6)(5)(4)}{6!}$$

$$= 128 + 448x + 672x^2 + 560x^3 + 280x^4 + 84x^5 + 14x^6 + x^7$$

Example:

Expand $\frac{1}{(1+2x)^3}$ in ascending powers of x as far as the term in x^3,using the binomial series.

Solution

Using the binomial expansion of $(1+x)^n$,where $n=-3$ and x is replaced by $2x$ gives:

$$\frac{1}{(1+2x)^3} = (1+2x)^{-3}$$

$$= 1 + (-3)(2x) + \frac{(-3)(-4)}{2!}(2x)^2 + \frac{(-3)(-4)(-5)}{3!}(2x)^3 + ...$$

$$= 1 - 6x + 24x^2 - 80x^3 + ...$$

The expansion is valid provided $|2x| < 1$

i.e $|x| < \frac{1}{2}$ or $-\frac{1}{2} < x < \frac{1}{2}$

Example:

Expand $\frac{1}{\sqrt{1-2t}}$ in ascending power of t as far as the term in t^3.State the limit of t for which the expression is valid.

Solution

$$\frac{1}{\sqrt{1-2t}} = (1-2t)^{-\frac{1}{2}}$$

$$= 1 + (-\frac{1}{2})(-2t) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-2t)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(-2t)^3 + ...$$

Using the expansion for $(1+x)^n$

$$= 1 + t + \frac{3}{2}t^2 + \frac{5}{2}t^3 + ...$$

The expansion is valid when $|2t| < 1$, i.e $|t| < \frac{1}{2}$ or $-\frac{1}{2} < t < \frac{1}{2}$

Example:

Express $\frac{\sqrt{1+2x}}{\sqrt[3]{1-3x}}$ as a power series as far as the term in x^2.State the range of values of x for which the series is convergent.

Solution:

$$\frac{\sqrt{1+2x}}{\sqrt[3]{1-3x}} = (1+2x)^{\frac{1}{2}}(1-3x)^{-\frac{1}{3}}$$

$$(1=2x)^{\frac{1}{2}} = 1 + (\frac{1}{2})2x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(2x)^2 + ...$$

$$= 1 + x - \frac{x^2}{2} + ...$$

which is valid for $|2x| < 1$ i.e $|x| < \frac{1}{2}$.

$$(1-3x)^{-\frac{1}{3}} = 1 + (-\frac{1}{2})(-3x) + \frac{(-\frac{1}{3})(-\frac{4}{3})}{2!}(-3x)^2 + ...$$

$$= 1 + x + 2x^2 + ...$$

which is valid for $|3x| < 1$, i.e $|x| < \frac{1}{3}$

Hence

$$\frac{\sqrt{1+2x}}{\sqrt[3]{1-3x}} = (1+2x)^{\frac{1}{2}}(1-3x)^{-\frac{1}{3}}$$

$$= (1 + x - \frac{x^2}{2} + ...)(1 + x + 2x^2 + ...)$$

$$= 1 + x + 2x^2 + xx^2 - \frac{x^2}{2} + ...$$

neglecting terms of higher power than 2

$$1 + 2x + \frac{5}{2}x^2$$

The series is convergent if $-\frac{1}{3} < x < \frac{1}{3}$

Note:

1. Binomial theorem when n is a positive integer

   If a, b are real numbers and n is a positive integer,then

   $$(a+b)^n = a^n+na^{n-1}b+\frac{n(n-1)}{2!}a^{n-2}b^2+\frac{n(n-1)(n-2)}{3!}a^{n-3}b^3+...+b^n$$

   or more concisely in terms of the binomial coefficient

   $$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

   we have

   $$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$$

   where

   $$\binom{n}{0} = \binom{n}{n} = 1$$

2. General form of the binomial theorem when α is arbitrary real number

   If a and b are real numbers such that $|b/a| < 1$ and α is an arbitrary real number,then

   $$(a+b)^\alpha = a^\alpha(1+b/a)^\alpha = a^\alpha(1+\frac{\alpha}{1!}(\frac{b}{a})+\frac{\alpha(\alpha-1)}{2!}(\frac{b}{a})^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}(\frac{b}{a})^3+...)$$

   The series on the right only terminates after a finite number of terms if α is a positive integer in which case the result reduces to the one just given.If α is a negative integer,or a non integral real number,the expression on the right becomes an infinite series that diverges if $1 < |a| > 1$

Example

Expand $(3+x)^{-\frac{1}{2}}$ by the binomial theorem,stating for what values of x the series converges.

Solution

Setting $\frac{b}{a} = \frac{1}{3}x$ in the general form of the binomial theorem gives:

$$(3+x)^{-\frac{1}{2}} = 3^{-\frac{1}{2}}(1+\frac{1}{3}x)^{-\frac{1}{2}} = \frac{1}{\sqrt{3}}(1-\frac{1}{6}x+\frac{1}{24}x^2-\frac{5}{432}x^3+...)$$

The series only converges if $|\frac{1}{3}x| < 1$ and so it is converges provided $|x| < 3$.

READING LIST:

1. S. A. Ilori, D. O. A. Ajayi, Algebra textbook,Y-Books, 2000.

2. A. Jeffrey, Advance Engineering Mathematics, University of News castle-upon-Tyne,2002. 3. A. Jeffrey,Engineering Mathematics,5th edition, 2001.