COURSE CODE: **MTS 321 LECTURE NOTE**

COURSE TITLE: **COMPLEX ANALYSIS 1**

NUMBER OF UNIT: **03**

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Other Lecturers:

COURSE CONTENT:

COURSE REQUIREMENTS:

Function of a complex variable. Limits and continuity of function of a complex variable, analytic functions, complex integrations, Cauchy's integral. Derivative theorems. Taylor's and Laurent's theorems. Classification of singularities. Convergence of sequence and series of complex functions(including power series and characterization of analytic functions by power series). Isolated singularities and residues. Residue theorem of algebra. Principle of analytic continuation. Multiple valued functions and Riemann surfaces.

READING LIST:

REFERENCES

(1) **E. Kreyszig**: Advanced Engineering Mathematics, 8th edition, Published by John Wiley and sons, 1999 reprinted 2008.(more to be added)

LECTURE NOTES

FUNCTION OF A COMPLEX VARIABLE: In simple statement, complex analysis is concerned with complex functions that are differentiable in some domain. Therefore, we should first say what we mean by a complex function and then define the concepts of limit derivative in complex. The approach here is synonymous with that in calculus. A greater attention is required because it will show the fundamental differences between real and complex calculus.

Recall from calculus that a real function f defined on a set S of real numbers is a rule that assigns to every x in S a real number $f(x)$, called the value of f at x. However in complex, S is a set of complex numbers and a function f defined on S is a rule that assigns to every z in S a complex number w, called the value of f at z. Hence, we write

$$w = f(z)$$

Here z varies in S and is called a **complex variable**. The set S is called the **domain** of f.

Example: $w = f(z) = z^2 + 3z$ is a complex function defined for all z; that is, its domain S is the whole complex plane.

The set of all values of a function f is called the range of f.

w is complex, and we write $w = u + iv$, where u and v are the real and imaginary parts, respectively. Now w depends on $z = x + iy$. Hence u becomes a real function of x and y, and so does v. We may thus write

$$w = f(z) = u(x, y) + iv(x, y)$$

This shows that a complex function $f(z)$ is equivalent to a pair of real functions $u(x, y)$ and $v(x, y)$, each depending on the two real variables x and y.

EXAMPLE 1 Function of a complex variable

Let $w = f(z) = z^2 + 3z$. Find u and v and calculate the value of f at $z = 1 + 3i$.

SOLUTION: $u = \text{Re}\, f(z) = x^2 - y^2 + 3x$ and $v = 2xy + 3y$. Also,

This shows that $u(1,3) = -5$ and $v(1,3) = 15$. Check this by using the expression for u and v.

EXAMPLE 2 Function of a complex variable

Let $w = f(z) = 2iz + 6\bar{z}$. Find u and v and the value of f at $z = \frac{1}{2} + 4i$.

SOLUTION: $f(t) = 2i(x+iy) + 6(x-iy)$ gives $u(x,y) = 6x - 2y$ and $v(x,y) = 2x - 6y$. Also,

$$f(\frac{1}{2} + 4i) = 2i(\frac{1}{2} + 4i) + 6(\frac{1}{2} - 4i) = i - 8 + 3 - 24i = -5 - 23i$$

Check this as in Example 1

1.2 LIMIT AND CONTINUITY

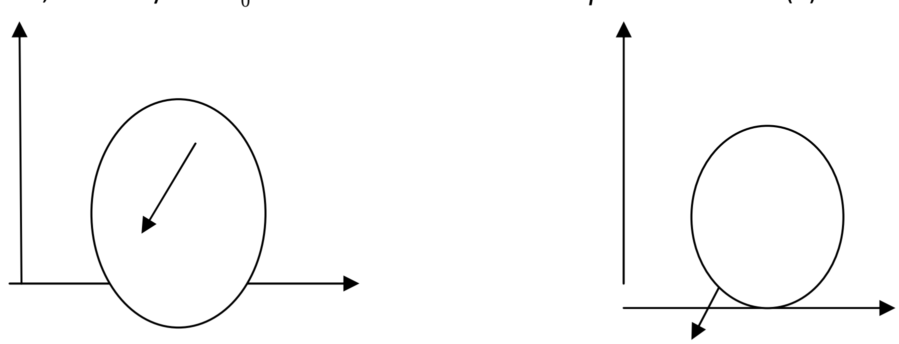
A function $f(z)$ is said to have the limit l as z approaches a point $z_{0,}$ written

(1) $\lim_{z \to z_0} f(z) = l,$

If f is defined in a neighborhood of (except perhaps at z_0 itself) and if the values of f are "close" to l for all z "close" to z_0; that is, in precise terms, for every positive real $\in$ we can find a positive real δ such that for all $z \neq z_0$ in the $|\ z - z_0\ | < \delta$ (Fig)

(2) $|\ f(z) - l\ | < \in;$

That is, for every $z \neq z_0$ in that δ disk the value of f lies in the disk (2).



Limit

Formally, this definition is similar to that in calculus, but there is a big difference. Whereas in the real case, x can approach an x_0 only along the real line, here, by definition, z may approach z_0 from direction in the complex plane. This will be quite essential in what follows.

If a limit exists, it is unique

A function $f(z)$ is said to be **continuous** at $z = z_0$ if $f(z_0)$ is define and

$$\lim_{z \to z_0} f(z) = f(z_0).$$

(3) Note that by the definition of a limit implies that $f(z)$ is defined in some neighborhood of z_0.

$f(z)$ is said to be continuous in a domain if it is continuous at each point of domain.

DERIVATIVE

The **derivative** of a complex function f at a point z_0 is written $f'(z_0)$ and is defined by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

provided this limit exists. Then f is said to be differentiable at Δz If we write $\Delta z = z - z_0$, and $z = z_0 + \Delta z$ we have

$$(4') \qquad f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Now come an **important point.** Remember that, by the definition of limit, $f(z)$ is defined in a neighborhood of z_0 and z in $(4')$ may approach z_0 from any direction in the complex plane. Hence differentiability at z_0 means that, along whatever path z approaches z_0, the quotient in $(4')$ always approaches a certain value and all these values are equal. This is important and should be kept in mind.

Example 3 Differentiability. Derivative

The function $f(z) - z^2$ is differentiable for all z and has the derivate $f'(z) = 2z$ because

$$f'(z) \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z^2 + 2z\Delta z + (\Delta z) - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

The differentiation rules are the same as in real calculus, since their proofs are literally the same. Thus,

$$(cf)' = cf', \ (f+g)' = f' + g', \ (fg)' = f'g + fg', \ \left(\frac{f}{g}\right)' = \frac{f'g - fg}{g^2}$$

as well as the chain rule and the power of rule $(z^n)' = nz^{n-1}$ (n integer) hold. Also, if $f(z)$ is differentiable at $z_{0,}$ it is continuous at z_0.

Example 4 $\bar{z}$ not differentiable

It is important to note that there are many simple functions that do not have a derivative at any point. For instance, $f(x) = \bar{2} = x - iy$ is such a function. Indeed, if we write $\Delta z = \Delta x + \Delta y$, we have

(5) $$\frac{f'(z+\Delta z) - f(z)}{\Delta z} = \frac{(\overline{2+\Delta z}) - \bar{2}}{\Delta z} = \frac{\bar{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

If $\Delta y = 0$, this is $+1$. If $\Delta x = 0$, this is -1. Thus (5) approaches $+1$ along path I but -1 along path II. Hence, by definition, the limit of (5) of $\Delta z \to 0$ does not exist at any z.

The example just discussed may be surprising, but it merely illustrates that differentiability of a complex function is a rather severe requirement.

The idea of proof (approach from different directions) is basic and will be used again in the next session.

1.3 ANALYTIC FUNCTION

These are functions that are differentiable in some domain, so that we can do calculus in complex. **They are the main concern of complex analysis.** Their introduction is our main goal in this section.

Definition (Analyticity)

A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D. The function $f(z)$ is said to be analytic at a point $z = z_0$ in D if $f(z)$ is analytic in a neighborhood of z_0.

Also, by an **analytic function** we mean a function that is analytic in some domain.

Hence analytic of $f(z)$ at z_0 (including z_0 itself since, by definition, z_0 is a point of all its neighborhoods). This concept is motivated by the fact that it is of no practical interest if a function is differentiable merely at a single point z_0 but no throughout some neigbhborhood of z_0. Team Project 24 gives an example.

A more modern term for analytic in D is holomorphic in D.

Diagram

Example 5 Polynomials, rational functions

The nonnegative integer powers $1, z, z^2, \ldots$ are analytic in the entire complex plane, and so are polynomials that is, functions of the form

$$f(z) = c_0 + c_{1^z} + c_{2^{z^2}} + \ldots + c_n z^n \qquad (6)$$

Where $c_0, \ldots c_n$ are complex constants.

The quotient of two Polynomials $g(z)$ and $h(z)$,

$$f(z) = \frac{g(z)}{h(z)}, \qquad (7)$$

Is called a rational function. This f is analytic except at the points where $h(z) = 0$, here we assume that common factors of g and h have been cancelled.

1.4 <u>CAUCHY-RIEMANN EQUATIONS</u>; The Cauchy-Riemann equation are the most important equations and one of the pillars on which complex analysis rests. They provide a criterion for the analyticity of a complex function

$$w = f(z) = u(x, y) + iv(x, y) \qquad (8)$$

f is analytic in a domain D if and only if the first partial derivatives u and v satisfy the two Cauchy-Riemann equations

$$u_x = v_y \quad , \quad u_y = -v_x \qquad (9)$$

Everywhere in D. Here $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$,

<u>EXAMPLE</u>; $f(z) = z^2 = x^2 - y^2 + 2ixy$ is analytic for all z, and $u = x^2 - y^2$ and $v = 2xy$ satisfy (9) since $u_x = 2x = v_y$ and $u_y = -2y = -v_x$

<u>EXAMPLE</u>; $f(z) = z^3 = x^3 - 3xy^2 + i(\ 3x^2 y - y^3\ )$ is analytic for all z, and satisfy (9) since and $u_x = 3x^2 - 3y^2$, $v_y = 3x^2 - 3y^2$, $u_y = -6xy$, and $v_x = 6xy$

THEOREM 1.1; Let $f(z) = u(x,y) + iv(x,y)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself. Then at that point, the first-order partial derivatives of u and v exist and satisfy the Cauchy-Riemann equations (9). Hence, if $f(z)$ is analytic in a domain D, equation (9) must be satisfied at all points of D.

PROOF; To be presented during the lecture

1.5 COMPLEX INTEGRATION

Integration in the complex plane is important for two reasons;

1. In applications there occurs real integrals that can be evaluated by complex integration, whereas the usual methods of real integral calculus fail;
2. Some basic properties of analytic functions can be established by complex integration, but would be difficult to prove by other methods. The existence of higher derivatives of analytic function is a striking property of this type.

1.6 LINE INTEGRALS; Complex definite integrals are called **line integrals.** They are written as

$$\int_C f(z)dz \qquad (10)$$

Here, the integrand $f(z)$ is integrated over a given curve C in the complex plane, called the path of integration. Such a curve may be represented parametrically as

$$z(t) = x(t) + iy(t), \qquad \text{for} \qquad a \le t \le b \qquad (11)$$

The sense of increasing t is called the positive sense on C, while C is assumed to be a smooth curve, that is, C has a continuous and nonzero derivative $\dot{z} = \frac{dz}{dt}$ at each point. Geometrically, this means that C has a unique and continuously turning tangent.

Definition of the Complex line Integral : This is similar to the method in calculus. Let C be a smooth curve in the complex plane given by (11), and let $f(z)$ be a continuous function given at each point of C. The interval $a \le t \le b$ in (11) is subdivided by points which yields

$$t_0(=a), t_1, ..., t_{n-1}, t_n(=b)$$

Where $t_0 < t_1, ... < t_n$. This corresponds to subdivision of C by points

$z_0, z_1, \ldots\ldots\ldots, z_{n-1}, z_n (= Z)$

Where $z_j = z(t_j)$. On each portion of subdivision of C, we choose an arbitrary point, say, a point ξ_1 between z_0 and z_1 (that is $\xi_1 = z(t)$ where t satisfies $t_0 \leqq t \leqq t_1$), a point ξ_2 between z_1 and z_2, etc. Then, we form the sum

$$S_n = \sum_{m=1}^{n} f(\xi_m)\Delta z_m \qquad \text{where} \qquad \Delta z_m = z_m - z_{m-1} \qquad (12)$$

We do this for each $n = 2,3,\ldots$ in a completely independent manner, but so that the greatest $|\Delta t_m| = |t_m - t_{m-1}|$ approaches zero as $n \to \infty$. This implies that the greatest $|\Delta t_m|$ also approaches zero because it cannot exceed the length of the arc of C from z_{m-1} to z_m and the latter goes to zero since the arc length of the smooth curve C is a continuous function of t. The limit of the sequence of complex numbers $S_2, S_3, \ldots$ obtained is called the **line integral** (or simply the integral) of $f(z)$ over the oriented curve C.

Figure:

This curve C is called the **path of integration**. Hence, the line integral is denoted by

$$\int_C f(z)dz \qquad (13)$$

or by

$$\oint_C f(z)dz \qquad (14)$$

if C is a **closed path** . However, it is generally assumed that all paths of integration for complex line integrals are **piecewise smooth**, that is they consist of finitely many smooth curves joined end to end.

Three basic properties of the Complex line Integral:

1.Linearity: integration is a linear operation

$$\int_C [\, k_1 f_1(z) + k_2 f_2(z) \,] dz = k_1 \int_C f_1(z)dz + k_2 \int_C f_2(z)dz \qquad (15)$$

2.sense reversal: integrating over the same path

$$\int_{z_0}^{Z} f(z)dz = -\int_{Z}^{z_0} f(z)dz \qquad (16)$$

3.Partitioning of path:

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz \qquad (17)$$

The existence of the complex line integral

To show that the line integral $\int_C f(z)dz$ exist with the assumption that the complex function $f(z)$ is continuous and C is piecewise smooth.

Let $f(z) = u(x, y) + iv(x, y)$

We set $\varsigma = \xi_m + i\eta_m$ and $\Delta z = \Delta x_m + i\Delta y_m$, such that the expression in (12) becomes

$$S_n = \sum (u + iv)(\Delta x_m + i\Delta y_m) \qquad (18)$$

where