MTS 102 Lecture Note

Trigonometry

By

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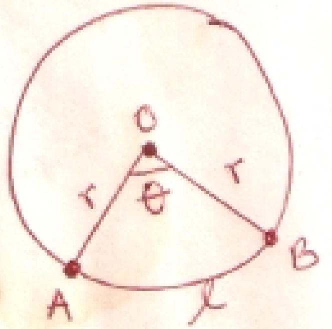
Circular Measure



In radian measure,

$l = r\theta$ — (1)

If $l = r$, then

$\theta = 1$ rad

From (1),

$\theta = \dfrac{l}{r}$

If $\theta = 180^\circ$, then

$180^\circ = \dfrac{\pi r}{r} = \pi$ rad — (2)

$\therefore 1^\circ = \dfrac{\pi}{180^\circ}$ rad

$30^\circ = \dfrac{\pi}{6}$, $60^\circ = \dfrac{\pi}{3}$,

$90^\circ = \dfrac{\pi}{2}$, $360^\circ = 2\pi$.

$$\text{Sector } AOB = \frac{\theta}{360^\circ} \times \pi r^2$$

$$= \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \theta.$$

EX Convert the following angles to radians:

(a) 45° (b) 120°, (c) 270° (d) 210°.

Solution: (a) $45^\circ = \frac{\pi}{180^\circ} \times 45^\circ$

$$= \frac{\pi}{4} \text{ rad.}$$

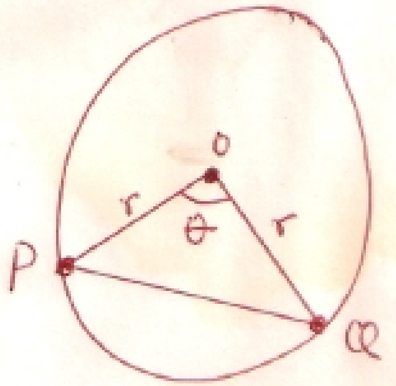
(d) $210^\circ = \frac{\pi}{180} \times 210^\circ$

$$= \frac{7\pi}{6} \text{ rad.}$$

EX A chord PQ of a ʘ with radius r subtends an angle θ at the centre. Show that the area of the minor segment PQ is $\frac{1}{2} r^2 (\theta - \sin\theta)$, and write down the area of the major segment

PQ in terms of r and θ.

Solution:



Sector POQ $= \frac{1}{2}r^2\theta$ — ①

Δ POQ $= \frac{1}{2}r^2\sin\theta$ — ②

subtracting ② from ①,

Area of minor seg PQ $= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

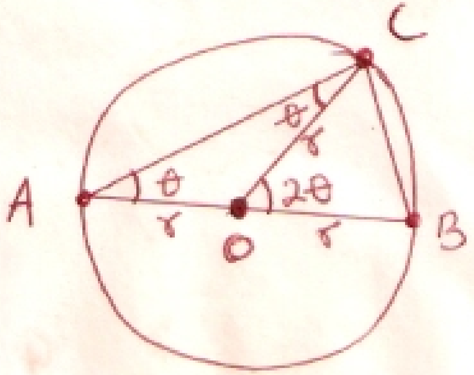
$$= \frac{1}{2}r^2\left[\theta - \sin\theta\right]$$

Area of major seg PQ = Area of ⊙ − Area of minor seg PQ

$$= \pi r^2 - \frac{1}{2}r^2\left[\theta - \sin\theta\right]$$

$$= \frac{1}{2}r^2\left[2\pi - \theta + \sin\theta\right].$$

EX



AB is a diameter of a ⊙ with radius r. C is a point on the circum. such that $C\hat{A}B = \theta$.

Show that the perimeter of the area bounded by CA, AB, and the arc BC is

$$2r\left[\theta + 1 + \cos\theta\right],$$

and find an expression for the area in terms of r and θ.

Solution: $A\hat{C}B = 90°$ [< in a semi ⊙]

$A\hat{C}O = \theta$ [base < of Iso Δ ACO]

$A\hat{O}C = 180° - 2\theta$ [Σ <s in a Δ]

$AC = 2r\cos\theta$ —— ①

$AB = 2r$ —— ②

$\text{Arc } BC = 2r\theta$ —— ③

Adding ①, ② and (3),

$AC + AB + \text{Arc } BC = 2r\cos\theta + 2r + 2r\theta$

$$= 2r\left[\theta + 1 + \cos\theta\right]$$

Which is the required perimeter.

$$\text{Sector } BOC = \frac{1}{2} r^2 \times 2\theta = r^2 \theta$$

$$\Delta AOC = \frac{1}{2} r^2 \sin(180^\circ - 2\theta)$$

$$= \frac{1}{2} r^2 \sin 2\theta$$

$$\therefore \text{Sector } BOC + \Delta AOC = r^2 \theta + \frac{1}{2} r^2 \sin 2\theta$$

$$= \frac{1}{2} r^2 \left[2\theta + \sin 2\theta \right]$$

Which is the required area.

Home Work

1. A chord AB subtends an angle 120° at O, the centre of a ⊙ with radius 12 cm. Find the area of:

   (a) sector AOB,

   (b) Δ AOB,

   (c) minor seg AB,

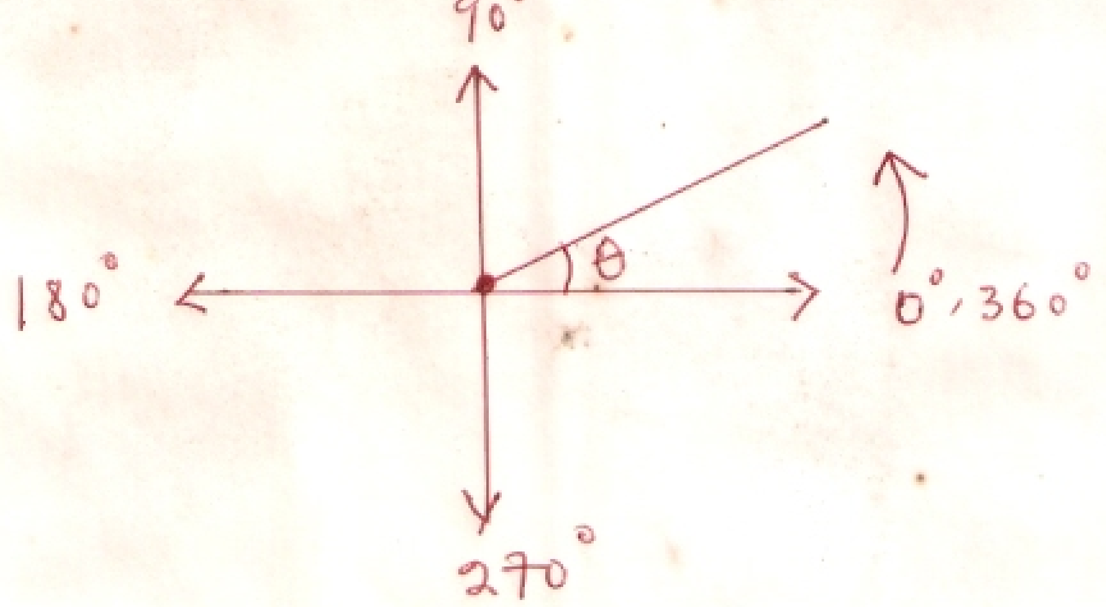
2. A ⊙ of radius r is drawn with its centre on the circumf. of another ⊙ of radius r.

Show that the area commm to both circle is

$$2r^2\left[\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right].$$

Trigonometry

Angles



Acute $0^\circ < \theta < 90^\circ$

Obtuse $90^\circ < \theta < 180^\circ$

Reflex $180^\circ < \theta < 360^\circ$.

Complimentary angles : $\theta_1 + \theta_2 = 90^\circ$.

Supplementary angles : $\theta_1 + \theta_2 = 180^\circ$.

Note Angles can be negative when measured in clockwise direction e.g. -20°, -60°, -210°.

To convert a negative angle to a positive angle, we add to the angle multiples of 360° i.e

$$K \times 360^\circ \quad \text{where} \quad K = 1, 2, 3, \ldots$$

Ex Convert the following to +ve angles:

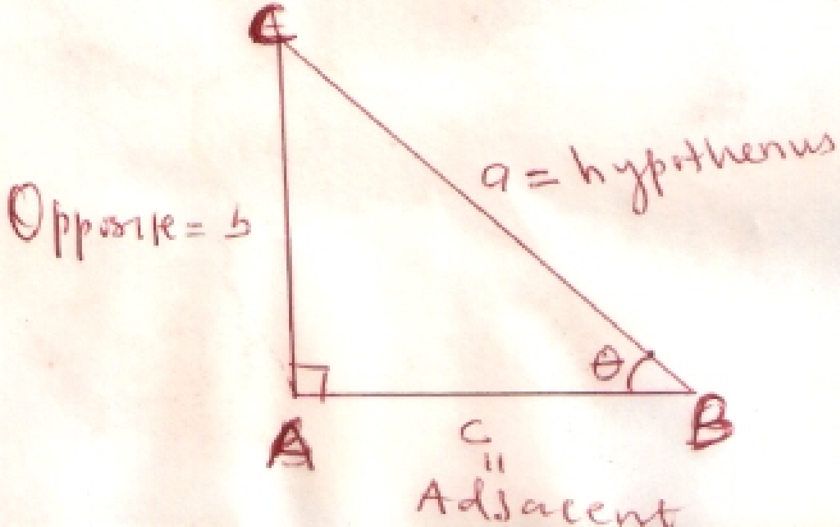
(i) -170° (ii) -520° (iii) -860°.

Solution: (i) $-170^\circ = -170^\circ \times 360^\circ = 190^\circ$.

(ii) $-520^\circ = -520 + 2 \times 360^\circ = 200^\circ$.

(iii) $-860^\circ = -860^\circ + 3 \times 360^\circ = 220^\circ$.

Trigonometrical Ratios



Consider ΔABC right-angled at A and let $A\hat{B}C = \theta$. With respect to angle θ, we define the following trig. ratios:

1. $\sin\theta = \frac{b}{a}$,

2. $\cos\theta = \frac{c}{a}$,

3. $\tan\theta = \frac{b}{c}$,

4. $\cot\theta = \frac{c}{b}$,

5. $\sec\theta = \frac{a}{c}$,

6. $\operatorname{cosec}\theta = \frac{b}{c}$.

The following should be noted:

1. $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{\cot\theta}$.

2. $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$.

3. $\sec\theta = \frac{1}{\cos\theta}$,

4. $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$.

5. $\sin^2\theta + \cos^2\theta = 1$.

Proof: $LHS = \sin^2\theta + \cos^2\theta$

$$= (\sin\theta)^2 + (\cos\theta)^2$$

$$= \left(\frac{b}{a}\right)^2 + \left(\frac{c}{a}\right)^2$$

$$= \frac{b^2 + c^2}{a^2}$$

$$= \frac{a^2}{a^2} \quad \text{[Pythagoras's theorem]}$$

$$= 1 = RHS.$$

Dividing through (5) by $\cos^2\theta$,

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\Rightarrow 1 + \operatorname{Tan}^2\theta = \sec^2\theta \quad \text{— (6)}$$

Also dividing through (5) by $\sin^2\theta$,

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\Rightarrow \quad 1 + \cot^2\theta = \operatorname{cosec}^2\theta \quad — \quad (7)$$

<u>Note</u> 1 — 7 are very useful identities and should be committed into memory.

<u>Note</u> All trig ratios are functions whose arguments can be either negative or positive angles e.g for $\sin\theta$, sin is a function and θ is its argument. We can have $\sin(-120^\circ)$, $\sin(30^\circ)$ etc.

It should be noted that sin, Tan, cot and cosec are odd functions but cos and sec are even functions i.e

1. $\sin(-\theta) = -\sin\theta$.

2. $\tan(-\theta) = -\tan\theta$.

3. $\cot(-\theta) = -\cot\theta$.

4. $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$.

5. $\cos(-\theta) = \cos\theta$.

6. $\sec(-\theta) = \sec\theta$.

<u>Note</u> If θ_1 and θ_2 are complimentary angles i.e $\theta_1 = 90^\circ - \theta_2$, then

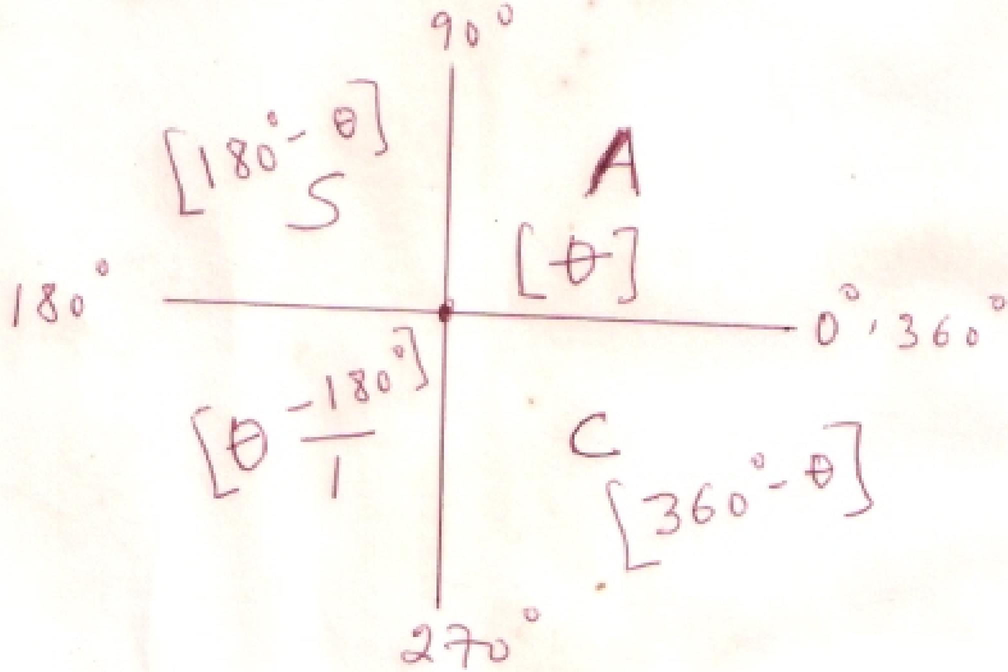
1. $\sin\theta_1 = \cos\theta_2$.

2. $\tan\theta_1 = \cot\theta_2$.

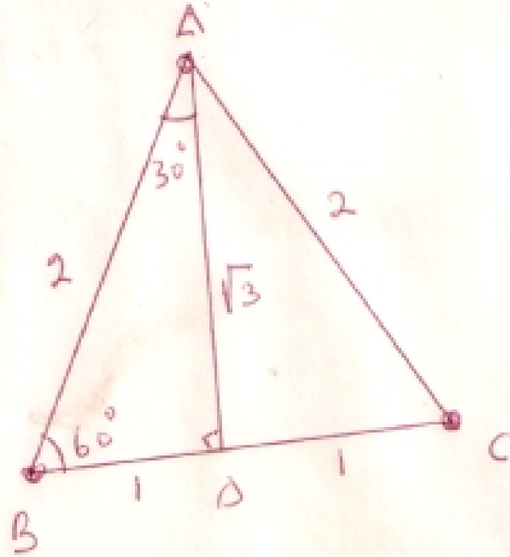
3. $\sec\theta_1 = \operatorname{cosec}\theta_2$.

Values of Trig Ratios for $\theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ, 180^\circ, 270^\circ$ and 360°.

For all values of $0° \le \theta \le 90°$, all trig ratios are positive. For $90° < \theta < 180°$, Only sin is +ve and others −ve. For $180° < \theta < 270°$, Only Tan is +ve, others are −ve. For $270° < \theta < 360°$, Only cos is +ve, Others are −ve.



Consider an equilateral Δ ABC of sides 2 units and let AD be the bisector of BC as shown



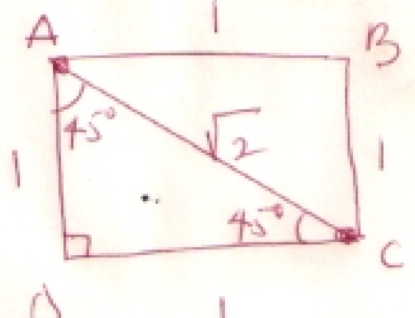
$$AD^2 = AB^2 - BD^2$$
$$= 4 - 1$$
$$= 3$$
$$\therefore AD = \sqrt{3}.$$

From ΔABD,

$\sin 30^\circ = 1/2$, $\cos \theta = \frac{\sqrt{3}}{2}$, $\tan 30^\circ = \frac{1}{\sqrt{3}}$.

$\sin 60^\circ = \frac{\sqrt{3}}{2}$, $\cos 60^\circ = 1/2$. $\tan 60^\circ = \sqrt{3}$.

consider a square of sides 1 unit and draw the diagonal AC as shown:



From ΔACD,

$$AC^2 = AD^2 + CD^2$$
$$= 1 + 1 = 2$$
$$\therefore AC = \sqrt{2}.$$

$\sin 45^\circ = \frac{1}{\sqrt{2}}$, $\cos 45^\circ = \frac{1}{\sqrt{2}}$, $\tan 45^\circ = 1$.

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$	$360°$
$\sin\theta$	0	$1/2$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	[illegible]	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$1/2$	0	-1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0
$\cot\theta$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	∞	0	∞
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-1	∞	1
$\operatorname{cosec}\theta$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	∞	-1	∞

This table is very important and every student must put it in the memory.

EX Evaluate the following:

(a) $\tan 120°$ (b) $\sin 240°$ (c) $\cos 305°$.

(d) $\cos(-30°)$ (e) $\tan(-60°)$ (f) $\sin(-210°)$

Solution:

(a) $\tan 120° = -\tan(180° - 120°)$ [2nd quadrant]

$= -\tan 60°$

$= -\sqrt{3}$.

(b) $\sin 240° = -\sin[240° - 180°]$ [3rd quadrant]

$= -\sin 60°$

$= -\dfrac{\sqrt{3}}{2}$.

(c) $\cos 305° = \cos(360° - 305°)$ (4th quadrant)

$= \cos 55°$

$= 0.5736$.

(d) $\cos(-30°) = \cos 30°$ [Even]

$= \dfrac{\sqrt{3}}{2}$.

(e) $\operatorname{Tan}(-60^\circ) = -\operatorname{Tan} 60^\circ$ [odd]

$= -\sqrt{3}$.

(f) $\sin(-210^\circ) = -\sin 210^\circ$ [odd]

$= (-1) \times (-1) \sin(210^\circ - 180^\circ)$ (3rd quadrant)

$= \sin 30^\circ$

$= 1/2$.

<u>EX</u> (a) The sine of an angle is -0.4591. The angle lies between 550° and 750° and its cosine is $-$ve. Find the angle.

(b) Find the Value of $\dfrac{\cos 330^\circ}{\operatorname{Tan} 210^\circ}$.

Solution: (a) Sine $-$ve and cosine $-$ve $\Rightarrow$ 3rd quadrant.

$\sin x = 0.4591$

$\Rightarrow x = 180^\circ + 27.32^\circ$

$= 207.32^\circ$

Since the angle must lie between 550° and 750°, the required angle $= 360^\circ + 207.32^\circ$

$= 567.32^\circ$.

(b) $$\frac{\cos 330^\circ}{\tan 210^\circ} = \frac{\cos(360^\circ - 330^\circ)}{\tan(210^\circ - 180^\circ)}$$

$$= \frac{\cos 30^\circ}{\tan 30^\circ} = \frac{\sqrt{3}/2}{1/\sqrt{3}}$$

$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{1} = \frac{3}{2}.$$

<u>EX</u> If $0^\circ \le x \le 360^\circ$, solve the following equations:

(a) $\cos x + \sin x = \dfrac{1}{\cos x - \sin x}$,

(b) $3\tan x + 4 = \dfrac{2}{\cos^2 x}$.

(c) $\sin x = 2\cos x$.

Solution: (a)

$$\cos x + \sin x = \frac{1}{\cos x - \sin x}$$

$$\Rightarrow (\cos x + \sin x)(\cos x - \sin x) = 1.$$

$$\Rightarrow \cos^2 x - \sin^2 x = 1$$

$$\Rightarrow 1 - \sin^2 x - \sin^2 x = 1$$

$$\Rightarrow -2\sin^2 x = 0$$

$$\Rightarrow \sin x = 0$$

$$\therefore x = 180^\circ \text{ or } 360^\circ.$$

(b) $$3\tan x + 4 = \frac{2}{\cos^2 x}$$

$$\Rightarrow 3\tan x + 4 = 2\sec^2 x$$

$$= 2\left[1 + \tan^2 x\right]$$

$$\Rightarrow 2\tan^2 x - 3\tan x - 2 = 0.$$

$$\Rightarrow (2\tan x + 1)(\tan x - 2) = 0$$

$$\therefore \tan x = -\frac{1}{2} \text{ or } \tan x = 2.$$

$\tan x = -\frac{1}{2}$

$\Rightarrow x = 180° + 26.6°$ or $360° - 26.6°$

$= 153.4°$ or $333.4°$.

$\tan x = 2$

$\Rightarrow x = 63.4°$ or $180° + 63.4° = 243.4°$.

(c) $\sin x = 2\cos x$

$\Rightarrow \frac{\sin x}{\cos x} = 2$

$\Rightarrow \tan x = 2$

$\therefore x = 63.4°$ or $243.4°$.

Home Work

1. Evaluate the following:

(a) $\tan 105°$ (b) $\sin(-500°)$

(c) $\cos(-75°)$ (d) $\sec(-120°)$

(e) $\cot(-30°)$ (f) $\operatorname{cosec}(-420°)$.

(2) The secant of an angle is 4.123. The angle is lying between 600° and 1000° and its cotangent is +ve, Find the angle.

3. If $0 \le x \le 360^\circ$, solve the following equations:

(a) $2\sin x = 1 + \frac{1}{\sin x}$.

(b) $\cot^2 x = 2 + \cot x$.

(c) $2\cot x + \tan x - 3 = 0$.

(d) $\sec^2 x - 3\tan x + 1 = 0$.

(e) $3\cos^2 x = 2\sin x \cos x$.

(f) $5\sin^2 x - 17\sin x + 6 = 0$.

(g) $2\cos^2 x + 3\cos x + 1 = 0$.

Compound Angles

If A and B are any two angles. then $\alpha = A+B$ and $\beta = A-B$ are well defined and we can find $\sin\alpha$, $\sin\beta$. $\cos\alpha$, $\cos\beta$, $\tan\alpha$, $\tan\beta$, $\cot\alpha$, $\cot\beta$. $\sec\alpha$, $\sec\beta$, $\operatorname{cosec}\alpha$ and $\operatorname{cosec}\beta$.

Theorem $\sin(A+B) = \sin A\cos B + \sin B\cos A$.

$\cos(A+B) = \cos A\cos B - \sin A\sin B$.

Task To find expressions for:

1. $\sin(A-B)$
2. $\cos(A-B)$
3. $\tan(A+B)$
4. $\tan(A-B)$

Solution: 1. We already know that

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

Put $B = -B$, we have

$$\sin(A+(-B)) = \sin(A-B)$$

$$= \sin A \cos(-B) + \cos A \sin(-B)$$

$$= \sin A \cos B - \cos A \sin B.$$

2. $\cos(A+B) = \cos A \cos B - \sin A \sin B$

put $B = -B$,

$$\cos(A+(-B)) = \cos(A-B)$$

$$= \cos A \cos(-B) - \sin A \sin(-B)$$

$$= \cos A \cos B + \sin A \sin B.$$

3. $\text{Tan}(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$= \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Now putting $B = -B$, we have

4. $\tan(A + (-B)) = \tan(A - B)$

$$= \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)}$$

$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$ [Tan is an odd function]

EX Find in surd form:

(a) $\sin 15^\circ$ (b) $\cos 105^\circ$ (c) $\tan 345^\circ$.

Solution: (a) $\sin 15^\circ = \sin(45^\circ - 30^\circ)$

$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$.

(b) $\cos 105^\circ = \cos(60^\circ + 45^\circ)$

$= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$

$= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$

$= \dfrac{1 - \sqrt{3}}{2\sqrt{2}}$

(c) $\text{Tan}(345^\circ) = \text{Tan}(360^\circ - 15^\circ)$

$= \dfrac{\text{Tan } 360^\circ - \text{Tan } 15^\circ}{1 + \text{Tan } 360^\circ \text{ Tan } 15^\circ}$

$= \dfrac{0 - \text{Tan } 15^\circ}{1 + 0 \times \text{Tan } 15^\circ}$

$= -\text{Tan } 15^\circ$

$= -\text{Tan}(45^\circ - 30^\circ)$

$= -\dfrac{\text{Tan } 45^\circ - \text{Tan } 30^\circ}{1 + \text{Tan } 45^\circ \text{ Tan } 30^\circ}$

$= \dfrac{\frac{1}{\sqrt{3}} - 1}{1 + \frac{1}{\sqrt{3}}}$

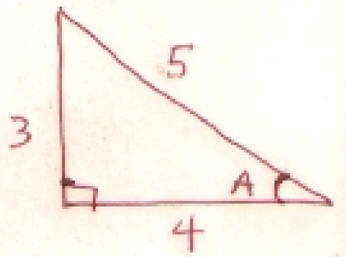
$$= \frac{1-\sqrt{3}}{1+\sqrt{3}}$$

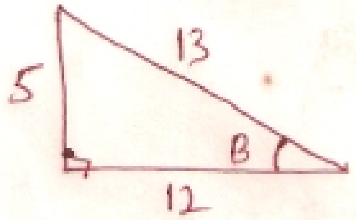
EX Given that $\cos A = 4/5$ and $\cos B = 12/13$, find the values of

(a) $\sin(A+B)$ (b) $\cos(A-B)$ (c) $\tan(A+B)$

If A and B are acute angles.

Solution:





$\sin A = 3/5$ $\sin B = 5/13$

$\tan A = 3/4$ $\tan B = 5/12$

(a) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$= \frac{3}{5} \times \frac{12}{13} + \frac{4}{5} \times \frac{5}{13}$$

$$= \frac{56}{65}.$$

(b) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$= \frac{4}{5} \times \frac{12}{13} + \frac{3}{5} \times \frac{5}{13} = \frac{63}{65}.$$

(c) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$$= \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \times \frac{5}{12}}$$

$$= \frac{14}{12} \times \frac{48}{33}$$

$$= \frac{56}{33}.$$

Home work

1. Find the following in surd form.

(a) $\sin 165^\circ$ (b) $\cos 195^\circ$ (c) $\cos 345^\circ$

2. If $180^\circ < A < 270^\circ$, $90^\circ < B < 180^\circ$ and $\tan A = 4/3$, $\cos B = -5/13$, find without using tables:

(a) $\cos(A-B)$ (b) $\tan(A+B)$ (c) $\sin(A-B)$

(d) $\sec(A+B)$ (e) $\cot(A-B)$ (f) $\operatorname{cosec}(A+B)$.

Double And Half Angles

Task To find expressions for:

(1) $\sin 2A$ (2) $\cos 2A$ (3) $\tan 2A$

(4) $\sin A$ (5) $\cos A$ (6) $\tan A$

Solution: (1) We know that

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

put $A = B$, to have

$$\sin(A+A) = \sin 2A$$
$$= \sin A \cos A + \cos A \sin A$$
$$= 2 \sin A \cos A.$$

(2) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

put $A = B$,

$$\cos(A+A) = \cos 2A$$
$$= \cos A \cos A - \sin A \sin A$$
$$= \cos^2 A - \sin^2 A$$
$$= \cos^2 A - \left[1 - \cos^2 A\right]$$

$$= 2\cos^2 A - 1 \quad \text{———} \quad (*)$$

$$= 2\left[1 - \sin^2 A\right] - 1$$

$$= 1 - 2\sin^2 A \quad \text{———} \quad (**)$$

(3) $\text{Tan}(A+B) = \dfrac{\text{Tan}A + \text{Tan}B}{1 - \text{Tan}A\,\text{Tan}B}$

put $A = B$,

$\text{Tan}(A+A) = \text{Tan}\,2A$

$$= \frac{\text{Tan}A + \text{Tan}A}{1 - \text{Tan}A\,\text{Tan}A}$$

$$= \frac{2\,\text{Tan}A}{1 - \text{Tan}^2 A}$$

(4) From (1),

$\sin 2A = 2\sin A\cos A$

Dividing all the angles by 2, we have

$$\sin A = 2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A.$$

(5) $\cos 2A = \cos^2 A - \sin^2 A$

$$= 2\cos^2 A - 1$$

$$= 1 - 2\sin^2 A$$

$\therefore \cos A = \cos^2\tfrac{1}{2}A - \sin^2\tfrac{1}{2}A$

$$= 2\cos^2\tfrac{1}{2}A - 1$$

$$= 1 - 2\sin^2\tfrac{1}{2}A.$$

(6) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

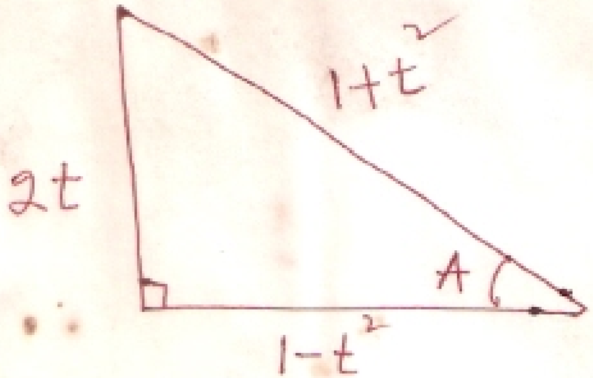
$\therefore \tan A = \dfrac{2\tan\frac{1}{2}A}{1-\tan^2\frac{1}{2}A}$

$$= \frac{2t}{1-t^2}$$

Where $t = \tan\tfrac{1}{2}A$.

Task: To find expressions for $\sin A$ and $\cos A$ in terms of t where $t = \tan \frac{1}{2} A$.

Solution:



$$\sin A = \frac{2t}{1+t^2}$$

$$\cos A = \frac{1-t^2}{1+t^2}.$$

EX (a) If $\tan \theta = 4/3$, find the possible values of $\tan \frac{1}{2}\theta$.

(b) Show that $\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1$.

(c) If $\cos A = 4/5$, find $\sin 2A$, $\cos \frac{1}{2}A$, and $\tan \frac{1}{2}A$.

(d) If $\cos 30^\circ = \frac{\sqrt{3}}{2}$, show that $\sin 15^\circ = \frac{1}{2}\sqrt{2-\sqrt{3}}$.

Solution: (a)

$$\tan\theta = \frac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}\theta}$$

$$\Rightarrow \frac{2t}{1-t^2} = \frac{4}{3}, \quad t = \tan\tfrac{1}{2}\theta$$

$$\Rightarrow 6t = 4 - 4t^2$$

$$\therefore 2t^2 + 3t + 2 = 0$$

$$2t^2 + 3t - 2 = 0$$

$$(2t-1)(t+2) = 0$$

$$\therefore t = 1/2 \text{ or } t = -2.$$

(b)

$$\tan 45^\circ = \frac{2\tan 22\frac{1}{2}^\circ}{1-\tan^2 22\frac{1}{2}^\circ}$$

$$\Rightarrow \frac{2t}{1-t^2} = 1, \quad t = \tan^2 22\tfrac{1}{2}^\circ$$

$$\Rightarrow 2t = 1 - t^2$$

$$\Rightarrow t^2 + 2t - 1 = 0$$

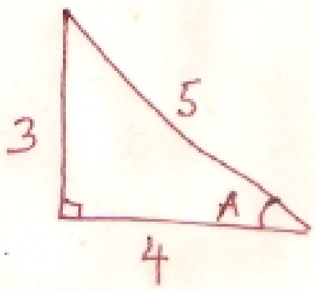
$$\therefore t = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2}$$

$$= -1 \pm \sqrt{2}$$

For an acute angle $22\frac{1}{2}^{\circ}$,

$$t = -1 + \sqrt{2}.$$

(c)



$\sin A = 3/5$,

$\tan A = 3/4$.

$$\sin 2A = 2 \sin A \cos A$$

$$= 2 \times \frac{3}{5} \times \frac{4}{5}$$

$$= \frac{24}{25}.$$

$$\cos A = 2\cos^2 \tfrac{1}{2}A - 1 = \frac{4}{5}$$

$$\Rightarrow \cos^2 \tfrac{1}{2}A = \frac{9}{10}$$

$$\therefore \cos \tfrac{1}{2}A = \frac{\pm 3}{\sqrt{10}}$$

$$= \frac{3}{\sqrt{10}}.$$

$$\cos A = \frac{1-t^2}{1+t^2}, \quad t = \tan \tfrac{1}{2}A.$$

$$\Rightarrow \frac{1-t^2}{1+t^2} = \frac{4}{5}$$

$$\Rightarrow 5-5t^2 = 4t^2+4$$

$$\Rightarrow 9t^2-1 = 0$$

$$\Rightarrow (3t+1)(3t-1)=0$$

$$\therefore \quad t = 1/3 .$$

(d) $\cos A = 1 - 2\sin^2 \frac{1}{2} A$

If $A = 30^\circ$, we have

$$1 - 2\sin^2 15^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\Rightarrow 2\sin^2 15^\circ = 1 - \frac{\sqrt{3}}{2} = \frac{2-\sqrt{3}}{2}$$

$$\Rightarrow \sin^2 15^\circ = \frac{2-\sqrt{3}}{4}$$

$$\therefore \sin 15^\circ = \frac{1}{2}\sqrt{2-\sqrt{3}} .$$

<u>EX</u> If θ is an acute angle and $\cos 2\theta = \frac{119}{169}$, find the values of $\sin\theta$ and $\cos\theta$.

Solution:

$$\cos 2\theta = 2\cos^2\theta - 1 = \frac{119}{169}$$

$$\Rightarrow \cos^2\theta = \frac{1 + \frac{119}{169}}{2}$$

$$= \frac{288}{2 \times 169} = \frac{144}{169}$$

$$\therefore \cos\theta = \frac{12}{13}.$$

Also,

$$\cos 2\theta = 1 - 2\sin^2\theta = \frac{119}{169}$$

$$\Rightarrow \sin^2\theta = \frac{1 - \frac{119}{169}}{2} = \frac{50}{2 \times 169}$$

$$= \frac{25}{169}$$

$$\therefore \sin\theta = \frac{5}{13}.$$

Home Work

1. Evaluate without tables:

(a) $2\cos^2\frac{\pi}{8} - 1$ (b) $\sin\frac{3\pi}{8}\cos\frac{3\pi}{8}$

(c) $\sin^2 22.5^\circ - \cos^2 22.5^\circ$

(d) $$\frac{2\tan\frac{3\pi}{8}}{1-\tan^2\frac{3\pi}{8}},$$

2. Given that $2\cos 36^\circ = 1 + 2\cos 72^\circ$, show that $4\cos^2 36^\circ - 2\cos 36^\circ - 1 = 0$ and hence deduce that

$$\cos 36^\circ = \frac{1}{4}\left(1+\sqrt{5}\right).$$

Tripple Angles

Task Find expressions for:

(a) $\sin 3A$ (b) $\cos 3A$ (c) $\tan 3A$.

Solution: (a) $\sin 3A = \sin(A+2A)$

$= \sin A \cos 2A + \cos A \sin 2A$

$= \sin A\left[1-2\sin^2 A\right] + \cos A \times 2\sin A\cos A$

$= \sin A - 2\sin^3 A + 2\sin A\left[1 - \sin^2 A\right]$

$= 3\sin A - 4\sin^3 A.$

(b) $\cos 3A = \cos(A + 2A)$

$= \cos A \cos 2A - \sin A \sin 2A$

$= \cos A\left[2\cos^2 A - 1\right] - \sin A \times 2\sin A \cos A$

$= \cos A\left[2\cos^2 A - 1\right] - 2\cos A\left[1 - \cos^2 A\right]$

$= 4\cos^3 A - 3\cos A.$

(c) $\tan 3A = \tan(A + 2A)$

$$= \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$$

$$= \frac{\tan A + \dfrac{2\tan A}{1 - \tan^2 A}}{1 - \tan A \times \dfrac{2\tan A}{1 - \tan^2 A}}$$

$$= \frac{\tan A - \tan^3 A + 2\tan A}{1 - \tan^2 A - 2\tan^2 A}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

If we put $A = 15^\circ$, then

$$\tan 45^\circ = \frac{3\tan 15^\circ - \tan^3 15^\circ}{1 - 3\tan^2 15^\circ} = 1$$

$$\Rightarrow \frac{3x - x^3}{1 - 3x^2} = 1, \quad x = \tan 15^\circ.$$

$$\Rightarrow 3x - x^3 = 1 - 3x^2$$

$$\Rightarrow x^3 - 3x^2 - 3x + 1 = 0.$$

$$\Rightarrow (x+1)(x^2 - 4x + 1) = 0$$

$$\Rightarrow x^2 - 4x + 1 = 0$$

$$\therefore x = 2 - \sqrt{3}$$

$$\therefore \tan 15^\circ = 2 - \sqrt{3}.$$

Home Work

1. Show that $\tan(A+B+C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan A \tan C - \tan B \tan C}$$

and deduce that

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

Given that $\tan 15^\circ = 2 - \sqrt{3}$, obtain in simplest surd form the value of $\tan 5^\circ$.

2. Find the value of $\tan 105^\circ$ in surd form. Hence find the value of $\tan 35^\circ$ in surd form.

Factor formulae

Task: To find expressions for:

(a) $\sin A \pm \sin B$ (b) $\cos A \pm \cos B$.

Solution: (a) Recall that

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \quad \text{—} \; (1)$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y \quad \text{—} \; (2)$$

Adding (1) and (2), to obtain

$$\sin(x+y) + \sin(x-y) = 2 \sin x \cos y \quad \text{—} \; (3)$$

If $A = x+y$ and $B = x-y$, then

$x = \frac{1}{2}(A+B)$, $y = \frac{1}{2}(A-B)$ and

(3) becomes

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B) \; \text{—} \; (4)$$

Sustracting (2) from (1), we have

$$\sin(x+y) - \sin(x-y) = 2 \cos x \sin y$$

$$\therefore \sin A - \sin B = 2 \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B) \; \text{—} \; (5)$$

(b) Recall that

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \quad \text{—} \; (6)$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y \quad \text{—} \; (7)$$

Adding (6) and (7), to have

$$\cos(x+y) + \cos(x-y) = 2\cos x \cos y$$

$$\therefore \cos A + \cos B = 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) \quad — (8)$$

Subtracting (7) from (6), we have

$$\cos(x+y) - \cos(x-y) = -2\sin x \sin y$$

$$\therefore \cos A - \cos B = -2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \quad — (9)$$

EX (a) Show that:

(i) $$\frac{\cos 70^\circ - \cos 50^\circ}{\sin 70^\circ - \sin 50^\circ} = -\sqrt{3}$$

(ii) $$\frac{\sin x + \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x} = \tan 3x$$

(iii) $4\cos x \cos\left(x + \frac{2}{3}\pi\right)\cos\left(x + \frac{4}{3}\pi\right) = \cos 3x$.

(b) If $\sin x + \sin y = \alpha$ and $\cos x + \cos y = \beta$, show that

$$\cos^2\tfrac{1}{2}(x-y) = \frac{1}{4}\left(\alpha^2 + \beta^2\right).$$

Solution: (a) (i)

$$\frac{\cos 70^\circ - \cos 50^\circ}{\sin 70^\circ - \sin 50^\circ} = \frac{-2\sin 60^\circ \sin 10^\circ}{2\cos 60^\circ \sin 10^\circ}$$

$$= -\tan 60^\circ$$

$$= -\sqrt{3}.$$

(ii) $$\frac{\sin x + \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x}$$

$$= \frac{\sin 3x + 2\sin 3x \cos 2x}{\cos 3x + 2\cos 3x \cos 2x}$$

$$= \frac{\sin 3x\left[1 + 2\cos 2x\right]}{\cos 3x\left[1+2\cos 2x\right]}$$

$$= \tan 3x.$$

(iii) $$4\cos x \cos\left(x + \frac{2}{3}\pi\right)\cos\left(x+\frac{4}{3}\pi\right)$$

$$= \left[2\cos x\right]\left[2\cos\left(x+\frac{2}{3}\pi\right)\cos\left(x+\frac{4}{3}\pi\right)\right]$$

$$= 2\cos x\left[\cos\left(2x+\frac{2\pi}{3}\right) + \cos\frac{2\pi}{3}\right]$$

$$= 2\cos x\left[\cos 2x \cos\frac{2\pi}{3} - \sin 2x \sin\frac{2\pi}{3} - \frac{1}{2}\right]$$

$$= 2\cos x\left[\cos 2x - \frac{1}{2}\right]$$

$$= 2\cos x\left[2\cos^2 x - 1 - \frac{1}{2}\right]$$

$$= 4\cos^3 x - 3\cos x$$

$$= \cos 3x.$$

(5) $\alpha = \sin x + \sin y$

$$= 2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y)$$

$$\therefore \alpha^2 = 4\sin^2\tfrac{1}{2}(x+y)\cos^2\tfrac{1}{2}(x-y) \quad \text{— (1)}$$

$\beta = \cos x + \cos y$

$$= 2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y)$$

$$\therefore \beta^2 = 4\cos^2\tfrac{1}{2}(x+y)\cos^2\tfrac{1}{2}(x-y) \quad \text{— (2)}$$

Adding (1) and (2), we have

$$\alpha^2 + \beta^2 = 4\cos^2\tfrac{1}{2}(x-y)\left[\sin^2\tfrac{1}{2}(x+y) + \cos^2\tfrac{1}{2}(x+y)\right]$$

$$= 4\cos^2\tfrac{1}{2}(x-y) \times 1$$

$$= 4\cos^2\tfrac{1}{2}(x-y)$$

$$\therefore \cos^2\tfrac{1}{2}(x-y) = \frac{1}{4}(\alpha^2+\beta^2).$$

Home work

1. If A, B, C are in arithmetic progression (AP), show that

$$\tan B = \frac{\sin A + \sin C}{\cos A + \cos C}.$$

2. Show that

$$\frac{\sin 2(\alpha+\beta) + \sin 2\alpha - \sin 2\beta}{\sin 2(\alpha+\beta) - \sin 2\alpha + \sin 2\beta} = \tan\alpha\cot\beta.$$

Establishing Given Trig. Identities

Note To establish / prove a given trigonometrical identities. It is easier and simpler to deal with the complex part of the identity or to re-arrange the given identity and deal with the complex part.

EX Show that:

(a) $\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{1+\sin\theta}{\cos\theta} = 2\sec\theta.$

(b) $\tan\left(x+\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}-x\right) = 1.$

(c) $\dfrac{1+\cos x+\cos 2x}{\sin x+\sin 2x} = \cot x$

(d) $\tan\frac{1}{2}(A-B)+\tan\frac{1}{2}(A+B) = \dfrac{2\sin A}{\cos A+\cos B}.$

Solution (a)

$$\text{LHS} = \frac{\cos\theta}{1+\sin\theta} + \frac{1+\sin\theta}{\cos\theta}$$

$$= \frac{\cos^2\theta + (1+\sin\theta)^2}{\cos\theta(1+\sin\theta)}$$

$$= \frac{\cos^2\theta + 1 + 2\sin\theta + \sin^2\theta}{\cos\theta(1+\sin\theta)}$$

$$= \frac{2[1+\sin\theta]}{\cos\theta(1+\sin\theta)}$$

$$= 2\sec\theta = \text{RHS}.$$

(b) $\text{LHS} = \tan\left(\frac{\pi}{4}+x\right)\tan\left(\frac{\pi}{4}-x\right)$

$$= \frac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4}\tan x} \times \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}$$

$$= \frac{(1+\tan x)(1-\tan x)}{(1-\tan x)(1+\tan x)}$$

$$= 1 = \text{RHS}.$$

(c) $LHS = \dfrac{1 + \cos x + \cos 2x}{\sin x + \sin 2x}$

$$= \frac{1 + \cos x + 2\cos^2 x - 1}{\sin x + 2\sin x \cos x}$$

$$= \frac{\cos x(1 + 2\cos x)}{\sin x(1 + 2\cos x)}$$

$$= \cot x = RHS$$

(d) $LHS = \tan \frac{1}{2}(A-B) + \tan \frac{1}{2}(A+B)$

$$= \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A-B)} + \frac{\sin \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A+B)}$$

$$= \frac{\cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B) + \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}$$

$$= \frac{\frac{1}{2}(\sin A - \sin B) + \frac{1}{2}(\sin A + \sin B)}{\frac{1}{2}(\cos A + \cos B)}$$

$$= \frac{2\sin A}{\cos A + \cos B}$$

$= RHS.$

Home work

Estaslish the following:

1. $\dfrac{1-\sin\theta+\cos\theta}{1-\sin\theta} = \dfrac{1+\sin\theta+\cos\theta}{\cos\theta}.$

2. $\dfrac{1+\sin\theta}{1-\sin\theta} = (\tan\theta+\sec\theta)^2.$

3. $\dfrac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta-\sin\theta} = \sec^2\theta.$

4. $\sec(A+B) = \dfrac{\sec A\sec B\operatorname{cosec}A\operatorname{cosec}B}{\operatorname{cosec}A\operatorname{cosec}B - \sec A\sec B}.$

5. $\dfrac{\cos(A-B)-\cos(A+B)}{\sin(A+B)+\sin(A-B)} = \tan B.$

6. $$\frac{\sin\alpha + \cos\alpha}{\sin(\theta-\alpha)\sin(\alpha-\beta)} + \frac{\sin\beta + \cos\beta}{\sin(\theta-\beta)\sin(\beta-\alpha)}$$

$$= \frac{\sin\theta + \cos\theta}{\sin(\theta-\alpha)\sin(\theta-\beta)}.$$

7. $$\frac{\cos A}{\sin B} - \frac{\sin A}{\cos B} = \frac{2\cos(A+B)}{\sin 2B}.$$

8. $$\frac{\sin 4A + \sin 2A}{\cos 4A + \cos 2A + 1} = \tan 2A.$$

9. $$\frac{\sin 2\theta + \cos 2\theta + 1}{\sin 2\theta - \cos 2\theta + 1} = \cot\theta.$$

10. $$\tan x = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}.$$

11. $$\frac{\sin A - \sin 2A + \sin 3A}{\cos A - \cos 2A + \cos 3A} = \tan 2A.$$

12. $\sin 4A \sin 5A + \sin 2A \sin 11A = \sin 7A \sin 6A.$

Inverse Trigonometric Functions

If $\sin\theta = 0.5$ and θ is acute, we find from table or calculator that $\theta = 30^\circ$. The function that acted on 0.5 to give 30° is the inverse of sin and it is denoted by $\sin^{-1}$. If θ is the argument, we write $\sin^{-1}\theta$. Similarly we have $\cos^{-1}\theta$, $\tan^{-1}\theta$, $\cot^{-1}\theta$, $\sec^{-1}\theta$ and $\operatorname{cosec}^{-1}\theta$.

<u>EX</u> If $\sec x = -2.7654$, find the possible values of x for $0 \le x \le 360^\circ$.

<u>Solution</u> $\sec x = -2.7654$

$$\Rightarrow \cos x = \frac{-1}{2.7654} = -0.3616$$

The angle x is lying either in 2nd or 3rd quadrant where cos is negative. Therefore

$$x = 180^\circ - \cos^{-1} 0.3616 \quad \text{or} \quad 180^\circ + \cos^{-1} 0.3616$$

$$= 180^\circ - 68.8^\circ \quad \text{or} \quad 180^\circ + 68.8^\circ$$

$$= 111.2^\circ \quad \text{or} \quad 248.8^\circ.$$

<u>Note</u> If $y = \sin^{-1} x$, then $x = \sin y$ and the following are ~~than~~ true for appropriate values of x:

1. $\sin(\sin^{-1} x) = \sin^{-1}(\sin x) = x$.
2. $\cos(\cos^{-1} x) = \cos^{-1}(\cos x) = x$.
3. $\tan(\tan^{-1} x) = \tan^{-1}(\tan x) = x$.
4. $\cot(\cot^{-1} x) = \cot^{-1}(\cot x) = x$.
5. $\sec(\sec^{-1} x) = \sec^{-1}(\sec x) = x$.
6. $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = \operatorname{cosec}^{-1}(\operatorname{cosec} x) = x$.

EX (a) Show that:

(i) $\sin^{-1}\frac{1}{2} = \cos^{-1}\frac{\sqrt{3}}{2}$,

(ii) $2\,\mathrm{Tan}^{-1}x = \mathrm{Tan}^{-1}\frac{2x}{1-x^2}$,

(iii) $2\sin^{-1}\frac{5}{13} = \mathrm{Tan}^{-1}\frac{120}{119}$.

(b) Find x If

$\mathrm{Tan}^{-1}x + \mathrm{Tan}^{-1}(1-x) = \mathrm{Tan}^{-1}\frac{4}{3}$.

Solution: (a) (i) Let

$x = \sin^{-1}\frac{1}{2}$ and $y = \cos^{-1}\frac{\sqrt{3}}{2}$

$\Rightarrow \sin x = 1/2$ and $\cos y = \frac{\sqrt{3}}{2}$

$\therefore x = 30^\circ$ and $y = 30^\circ$

$\therefore x = y$.

(ii) Let $p = \mathrm{Tan}^{-1}x$ and $q = \mathrm{Tan}^{-1}\frac{2x}{1-x^2}$.

Then $x = \mathrm{Tan}\,p$ ——— (1)

$$\frac{2x}{1-x^2} = \tan q \quad \text{——— (2)}$$

and we must show that $2p = q$.

Using (1) ~~and~~ in (2), we have

$$\frac{2\tan p}{1-\tan^2 p} = \tan q$$

$$\Rightarrow \tan 2p = \tan q$$

$$\Rightarrow \tan^{-1}(\tan 2p) = \tan^{-1}(\tan q)$$

$$\Rightarrow 2p = q$$

$$\therefore \quad 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}.$$

(iii) Let $x = \sin^{-1}\frac{5}{13}$ and $y = \tan^{-1}\frac{120}{119}$.

Then $\sin x = \frac{5}{13}$ and $\tan y = \frac{120}{119}$

so that $\cos x = \frac{12}{13}$, $\tan x = \frac{5}{12}$,

$\sin y = \frac{120}{169}$ and $\cos y = \frac{119}{169}$.

Now, we can either prove that

case 1: $2 \sin^{-1} \frac{5}{13} = \tan^{-1} \frac{120}{119} = \sin^{-1} \frac{120}{169}$

or

case 2: $2 \sin^{-1} \frac{5}{13} = \tan^{-1} \frac{5}{12} = \tan^{-1} \frac{120}{119}$.

In the case 1:

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

$$= \frac{2 \times \frac{5}{12}}{1 - \frac{25}{144}} = \frac{120}{119} = \tan y$$

$\therefore 2x = y$

$\therefore 2 \sin^{-1} \frac{5}{13} = \tan^{-1} \frac{120}{119}$.

For case 2;

$\sin 2x = 2 \sin x \cos x$

$$= 2 \times \frac{5}{13} \times \frac{12}{13} = \frac{120}{169} = \sin y$$

$$\therefore \quad 2x = y$$

$$\therefore \quad 2\sin^{-1}\frac{5}{13} = \tan^{-1}\frac{120}{119}.$$

(b) Let $a = \tan^{-1}x$, $b = \tan^{-1}(1-x)$,

$c = \tan^{-1}\frac{4}{3}$ so that

$x = \tan a$, $1-x = \tan b$ and

$\frac{4}{3} = \tan c$. Now

$$a + b = c$$

$$\Rightarrow \tan(a+b) = \tan c$$

$$\Rightarrow \frac{\tan a + \tan b}{1 - \tan a \tan b} = \tan c$$

$$\Rightarrow \frac{x + (1-x)}{1 - x(1-x)} = \frac{4}{3}$$

$$\Rightarrow \frac{1}{x^2 - x + 1} = \frac{4}{3}$$

$$\Rightarrow \quad 4x^2 - 4x + 1 = 0$$

$$\Rightarrow \quad (2x-1)^2 = 0$$

$$\therefore \quad x = 1/2 .$$

Home work

1. Without using tables or calculator, show that:

(a) $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \tan^{-1}\frac{1}{2}$.

(b) $\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{63}{65} = 2\tan^{-1}\frac{1}{5}$.

(c) $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4}$.

2. (a) Show that

$$\tan^{-1}\alpha + \tan^{-1}\beta = \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right).$$

(b) Find x if $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$.

Trigonometrical Equations

Note We have seen that if

$\sin x = 1/2$ and $0^\circ \leq x \leq 360^\circ$, then

$x = 30^\circ$ or 150°. We have

the following:

1. If $\sin x = \alpha$, then

$$x = (-1)^n \sin^{-1}\alpha + n\pi, \quad n = 0, 1, 2, \ldots$$

2. If $\cos x = \alpha$, then

$$x = 2n\pi \pm \cos^{-1}\alpha, \quad n = 0, 1, 2, \ldots$$

3. If $\tan x = \alpha$, then

$$x = n\pi + \tan^{-1}\alpha, \quad n = 0, 1, 2, \ldots$$

These are called the general solutions.

EX Solve for x generally and for $0^\circ \le x \le 360^\circ$ in the equation

$$\cos 2x = \sin 3x.$$

solution: $\sin 3x = \cos 2x$

$$\Rightarrow \quad \sin 3x = \sin(90^\circ - 2x)$$

$$\Rightarrow \quad \sin 3x - \sin(90^\circ - 2x) = 0$$

$$\Rightarrow \quad 2\cos\tfrac{1}{2}(x+90^\circ)\,\sin\tfrac{1}{2}(5x-90^\circ) = 0$$

$\therefore \cos\frac{1}{2}(x+90^\circ) = 0$ or

$\sin\frac{1}{2}(5x-90^\circ) = 0.$

When $\cos\frac{1}{2}(x+90^\circ) = 0$, then

$$\frac{1}{2}(x+90^\circ) = 2n\pi \pm \cos^{-1}0$$

$$= 2n\pi \pm 90^\circ$$

$$\therefore \quad x = 2\left[2n\pi \pm 90^\circ\right] - 90^\circ \quad \text{—} \ (1)$$

When $\sin\frac{1}{2}(5x-90^\circ) = 0$, then

$$\frac{1}{2}(5x - 90^\circ) = n\pi + (-1)^n \sin^{-1} 0$$

$$= n\pi$$

$$\therefore x = \frac{2n\pi + 90^\circ}{5} \quad \text{—— (2)}$$

From (2), we obtain:

$n = 0$: $x = 18^\circ$,

$n = 1$: $x = 90^\circ$,

$n = 2$: $x = 162^\circ$,

$n = 3$: $x = 234$,

$n = 4$: $x = 306$.

EX Solve the equation

$$4\cos^2 x + 5\sin^2 x = 5.$$

Solution: $4\cos^2 x + 5\sin^2 x = 5$

$\Rightarrow 4(1-\sin^2 x) + 5\sin^2 x = 5$

$\Rightarrow 4 - 4\sin^2 x + 5\sin^2 x = 5$

$\Rightarrow \sin^2 x - 1 = 0$

$\Rightarrow (\sin x - 1)(\sin x + 1) = 0$

$\therefore \sin x = 1$ or $\sin x = -1$

When $\sin x = 1$, then

$x = n\pi + (-1)^n \sin^{-1} 1$

$= n\pi + 90^\circ \times (-1)^n$, ① $n = 0, 1, 2, \ldots$

When $\sin x = -1$, then

$x = n\pi + (-1)^n \sin^{-1}(-1)$

$= n\pi + 270^\circ \times (-1)^n$, ② $0, 1, 2, \ldots$

From (2),

$n = 0$; $x = 270^\circ$.

$n=1$: $x = 180° - 270°$

$= -90° = 270°$.

From (1),

$n=0$: $x = 90°$.

$n=1$: $180-90 = 90°$

$\therefore x = 90°$ or $270°$.

Ex Solve for x if

$2\cos 3x \cos x = \cos 2x + \sin 2x + 1$.

Solution:

$2\cos 3x \cos x = \cos 2x + \sin 2x + 1$

$\Rightarrow \cos 4x + \cos 2x = \cos 2x + \sin 2x + 1$

$\Rightarrow \cos 4x = \sin 2x + 1$

$\Rightarrow 1 - 2\sin^2 2x = \sin 2x + 1$

$\Rightarrow 2\sin^2 2x + \sin 2x = 0$

$$\Rightarrow \sin 2x(2\sin 2x + 1) = 0$$

$$\therefore \quad \sin 2x = 0 \quad \text{or} \quad \sin 2x = -\tfrac{1}{2}.$$

When $\sin 2x = 0$, then

$$2x = n\pi + (-1)^n \sin^{-1} 0$$

$$\therefore \quad x = \frac{n\pi}{2}, \quad n = 0, 1, 2, \ldots \quad (1)$$

When $\sin 2x = -\tfrac{1}{2}$, then

$$2x = n\pi + (-1)^n \sin^{-1}(-\tfrac{1}{2})$$

$$= n\pi + 210^\circ \times (-1)^n.$$

$$\therefore \quad x = \frac{n\pi}{2} + 105^\circ \times (-1)^n, \quad n = 0, 1, 2 \ldots \quad (2)$$

From (1),

$n = 0$: $\quad x = 0^\circ$.

$n = 1$: $\quad x = 90^\circ$.

$n = 2$: $\quad x = 180^\circ$.

$n = 3$: $x = 270^\circ$.

$n = 4$: $x = 360^\circ$

From (2),

$n = 0$: $x = 105^\circ$.

$n = 1$: $x = 345$

$n = 2$: $x = 285^\circ$

$n = 3$: $x = 165^\circ$

$\therefore x = \{0, 90^\circ, 180^\circ, 270^\circ, 360^\circ, 105^\circ, 345^\circ, 285^\circ, 165^\circ\}$.

Home work

Find the general solutions and the values of x for $0^\circ \le x \le 360^\circ$ in the following equations.

1. $\sin 4x = \sin 2x$.
2. $\sin 2x \cos 3x = 0$.
3. $\tan 2x = \cot 3x$.

4. $4\cos x = 3\tan x + 3\sec x$.

5. $9\sin^2 x + 10\sin x\cos x - 2\cos^2 x = 1$.

6. $3\sin 3x - \operatorname{cosec} 3x + 2 = 0$.

7. $\tan 2x \tan 4x = 1$.

8. $\cos\frac{x}{2} + 2\cos\frac{3x}{2} + \cos\frac{5x}{2} = 0$.

9. $\sin x - 2\sin 2x + \sin 3x = 0$.

10. $\cos x + \cos 3x + \cos 5x + \cos 7x = 0$.

Solutions of the Equations of The form $a\cos x \pm b\sin x = c$

Consider the equation

$$a\cos x + b\sin x = c \quad \text{—— (1)}$$

Suppose the LHS of (1) can be expressed in the form

$$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha \quad \text{—— (2)}$$

Where R and α are unknowns to

be determined.

Comparing (1) and (2), we have

$$R\sin\alpha = a \quad \text{——} \quad (3)$$

$$R\cos\alpha = b \quad \text{——} \quad (4)$$

Squaring and adding (3) and (4),

$$R^2 = a^2 + b^2$$

$$\therefore R = \sqrt{a^2+b^2} \quad \text{——} \quad (5)$$

Dividing (3) by (4),

$$\text{Tan}\,\alpha = a/b$$

$$\therefore \alpha = 180^\circ n + \text{Tan}^{-1}(a/b), \quad n = 0, 1, 2, \ldots \quad \text{——} \quad (6)$$

Using (2) in (1), we have

$$R\sin(x+\alpha) \equiv LHS \quad \text{——} \quad (7)$$

From which we can determine

the maximum and minimum values of the LHS of (1).

EX Solve the equation $3\cos x + 4\sin x = 2$ for $0^\circ \le x \le 360^\circ$. State the max and min values of $3\cos x + 4\sin x$ and the values of x for which they occur.

Solution: suppose that

$$R\cos(x-\alpha) \equiv 3\cos x + 4\sin x$$

$$\Rightarrow R\cos x\cos\alpha + R\sin x\sin\alpha \equiv 3\cos x + 4\sin x$$

$$\Rightarrow \begin{aligned} R\sin\alpha &= 4 \\ R\cos\alpha &= 3 \end{aligned}$$

$$\therefore R^2 = 4^2 + 3^2 = 25$$

$$\therefore R = 5.$$

$\tan\alpha = \frac{4}{3} = 1.3333$

$\therefore \quad \alpha = 53.13^\circ$

$\therefore \quad 5\cos(x - 53.13^\circ) = 2$

$\Rightarrow x - 53.13^\circ = 2n\pi \pm \cos^{-1} 2/5$

$= 2n\pi \pm 66.42^\circ$

$\therefore \quad x = [2n\pi \pm 66.42^\circ] + 53.13^\circ$

$n = 0: \quad x = 119.55^\circ$

$n = 1; \quad x = 346.71^\circ.$

Lastly,

$3\cos x + 4\sin x \equiv 5\cos(x - 53.13)$

Since the max value of $\cos x$ is 1 and min value is -1 when $x = 0^\circ$ and 180° respectively. it follows that max value of the expression is 5 which occurs when

$$x - 53.13^\circ = 0$$

$$\therefore \quad x = 53.13^\circ.$$

The min value of the expression is -5 which occurs when

$$x - 53.13^\circ = 180^\circ$$

$$\therefore \quad x = 233.13^\circ.$$

Home work

1. Solve the following equations for $0^\circ \leq x \leq 360^\circ$:

(a) $5 \sin x - 12 \cos x = 6$.

(b) $4 \cos x \sin x + 15 \cos 2x = 10$.

(c) $3 \tan x - 2 \sec x = 4$.

(d) $\cos x + \sin x = \sec x$.

2. Find the max and min values

of the following expressions and state the values of x in $0° \leq x \leq 360°$ for which they occur.

(a) $8\cos x - 15\sin x$.

(b) $\dfrac{1}{\cos x - \sin x}$.

(c) $(2\cos x + 3\sin x)^2$.

(d) $\dfrac{1}{(\sin x - 2\cos x)^2}$.

(e) $5\cos 2x - \sqrt{2}\sin 2x$.

(f) $5\cos x + 12\sin x$.

References

1. Bunday B.D. and Mulholland H., Pure Maths for Advanced Level. (1988)

2. Backhouse J.K. and Houldsworth S.P.T., Pure Maths A First Course. (1982)