UNIVERSITY OF AGRICULTURE, ABEOKUTA

Department of Mathematics,

B.Sc Degree Examinations 2009/2010 First Semester

MTS 423, Functional Analysis,

Time Allowed: 3Hours. Attempt Any 4 Questions

Question 1

(a) Let $C([0,1])$ denote the space of continuous real valued functions on the interval $[0,1] \subseteq I\!R$.

Show that $C([0,1])$ is a normed space when equipped with the sup norm:

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|, \quad \forall f \in C([0,1]).$$

(b) Let X be a linear space. For each pair $(x,y) \in X \times X$, define a map $A : X \times X \to X \times X$ by

$$A(x,y) = (x + 2y, x - 2y).$$

(i) Is A linear ?

(ii) Find the inverse map A^{-1} if it exists.

(c) Let X and Y be normed spaces. When do we say that a linear map $T : D(T) \subseteq X \to Y$ is bounded ?

Let $X = Y = C([0,1])$. Show that the integral operator $T : X \to Y$ defined by :

$$(TX)(t) = \int_0^1 K(t,s)X(s)ds$$

is a bounded linear operator, where

$$K : [0,1] \times [0,1] \to I\!R$$

is a continuous function and $X \in C([0,1])$.

Question 2

(a) Show that all linear maps from a finite dimensional normed space are automatically continuous.

(b) (i) When do we say that a normed space is Banach?

(ii) Let X be the linear space of all polynomials on the interval $(0,1)$ with values in $I\!R$.

Define $\|\cdot\|$ on X by

$$\|x\| = \sup_{t\in(0,1)}, \quad x \in X.$$

Show that $(X, \|\cdot\|)$ is an incomplete normed space.

(c) Show that the space $(C(a,b), \|\cdot\|)$ is a Banach space, where $C(a,b)$ is the linear space of real valued continuous functions on the interval (a,b), $a < b$, $a,b \in I\!R$ equipped with the sup norm.

Question 3

(a) Define an inner product space $(H, < \cdot, \cdot >)$.
(b) For each $x \in H$, show that $\|x\| = \sqrt{< x, x >}$ is a norm on H.
(c) Let H be a pre-Hilbert space. Prove the Cauchy-Schwartz inequality:

$$| < x, y > | \leq \|x\| \|y\|, \ \forall x, y \in H.$$

(d) Equip $C(0,1)$ with the supremum norm and let $T = \frac{d}{dt}$.
(i) What is the natural domain of T in $C(0,1)$?
(ii) Show that T is linear.

Question 4

(a) Let X and Y be normed spaces and $T : X \to Y$. When is T said to be continuous at a point x_0 in X ? (b) If $X = C[0,1]$ equipped with the sup norm, and $Y = \mathbb{R}$, show that if X_0 is a fixed member of $C[0,1]$, then the map

$$H(f) = \int_0^1 2X_0(t) f(t) dt$$

is a bounded linear transformation from $C[0,1]$ to $\mathbb{R}$.
(c) Let H be a Hilbert space with inner product $< \cdot, \cdot >$ and let $X_0, \ Y_0 \in H$ be fixed. Define $X_0 \otimes Y_0 : H \to H$ by

$$(X_0 \otimes Y_0)(x) = < x, Y_0 > x_0, \ x \in H.$$

Show that $X_0 \otimes Y_0$ is bounded and linear on H and that

$$\|X_0 \otimes Y_0\| \leq \|X_0\| \|Y_0\|.$$

Question 5

(a) Prove or disprove the following statement: The Cartesian product of two Banach spaces X and Y is Banach.
(b) State and prove the Riesz Representation Theorem on a Hilbert space.
(c) Let (X, ρ) be a non empty complete metric space and $T : X \to X$, a contraction on X.
Show that T has a unique fixed point.

Dr E.O. Ayoola, June/July 2010.