UNIVERSITY OF AGRICULTURE, ABEOKUTA,

DEPARTMENT OF MATHLMATICS

Second Semester Examination , 2009/2010

MTS466- Calculus of variations

. INSTRUCTION: Answer Any Four Questions Time :2HRS

1 (a) Given a functional

$$I = \iiint F(U, U_x, U_y, U_z, x, y, z)\,dxdydz$$

Where $U_x = \frac{\partial U}{\partial x}$. Show that the Euler's equation for three independent variables can be written as $\frac{\partial F}{\partial U} - \frac{\partial}{\partial x}\frac{\partial F}{\partial U_x} - \frac{\partial}{\partial y}\frac{\partial F}{\partial U_y} - \frac{\partial}{\partial z}\frac{\partial F}{\partial U_Z} = 0$

(b) If an electrostatic field with energy density in terms of the static potential as $\frac{1}{2}\varepsilon(\nabla\Psi^2)$. Obtain the Laplace's equation of electrostatics $\nabla^2\Psi(x,y,z) = 0$

(c) Solve the problem of maximizing the function $f(x,y) = 2xy$ subject to $x^2 + y^2 - a^2 = 0$ using Lagrange's multiplier.

2. (a) .State the Hamilton's principle of motion ? . Given that $L = T - V$ where L is the Lagrangian, $T = \frac{1}{2}m\dot{x}^2$, V is the potential energy of the system and $F(x) = -\frac{dV(x)}{dx}$. Obtain the Newton's law of motion $F(x) = m\ddot{x}$.

(b) If the kinetic energy of a moving particle in cylindrical coordinate xy- plane is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}(\dot{\rho}^2 + \rho^2\dot{\Psi}^2)$$

Show that the radial acceleration and the statement of conservation of angular momentum are $\frac{d}{dt}(m\dot{\rho}) - m\rho\dot{\Psi}'^2 = 0$ and $\frac{d}{dt}(m\rho^2\dot{\Psi}) = 0$ respectively.

(c) Given that the length of the enclosed area of a plane is $\int_a^b \frac{1}{2}m(x\dot{y} - y\dot{x})dt$ subject to the constraint $\int_a^b \sqrt{\dot{x}^2 + \dot{y}^2}\,dt$. Show that $(y-c)^2 + (x+k)^2 = \lambda^2$

3 (a) Given a functional $I = \int_{x_1}^{x_2} F(x,y,y')dx$. Assuming the existence of an optimum path for which I is stationary, and a function $\eta(x)$ which defines the arbitrary deformation satisfying the boundary requirements that $\eta(x_1) = \eta(x_2) = 0$. Obtain the Euler's equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

(b) Suppose the element of distance describing an isoperimetric problem in Euclidean x-y plane given by $ds = \left[1 + (y')^2\right]^{\frac{1}{2}}$. obtain the equation $y = ax + b$.

(c) Solve the problem of two parallel coaxial wire circles to be connected by a surface of minimum area that is generated by resolving a curve $y(x)$ about the x- axis in which the curve is required to pass through the fixed end points. Such that the element of the area is given as

$$dA = 2\pi y\left[1+(y')^2\right]^{\frac{1}{2}}$$

4(a) Give and explain three examples of physical problems which are functional?.

(b) Prove that minimizing the functional

$$I[\phi] = \int_0^1 \left[\phi^2 + (\phi')^2\right]dx$$

With $\phi(0) = U_0$ and $\phi(1) = U_1$ is equivalent to solving a boundary value problem?.

(c) Deduce the boundary value problem equivalent to minimizing the functional

$$I[\phi] = \int_1^{31} \left[-(\phi')^2 - 4\phi\right]dx + 12\phi(3)$$

Where $\phi(1) = 1$.

5 (a) Define the following with reference good examples

(i) functional (ii) brachistochrone problem (iii) admissible functions

(b) Consider the quantum mechanical problem of a particle (mass m) in rectangular parallelepiped box with sides a, b, c. The ground state energy of the particle is given by

$$E = \frac{h^2}{8m}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right),$$

Subject to the constraint that the volume is constant, that is $V(a,b,c) = a\ b\ c = k$.

Show that $a = b = c$

(c) Suppose a nuclear reactor having a right-circular cylinder of radius R and height H with the volume of the reactor vessel

$$F(R,H) = \pi R^2 H$$

Subject to the Neutron diffusion theory constraint $\psi(R,H) = \left(\frac{2.4048}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2$. Solve for H in terms of R for the minimum volume of the reactor. .

6 (a) Write down the equations which forms the basis of Raleigh-Ritz method for the computation of eigenfunctions and eigenvalues?

(b) Consider a simple pendulum of mass m, constraint by a wire of length l swing in an arc.

$\psi_1 = r - l = 0$. With two generalized coordinates r and θ. Given that the Lagrangian is, for the potential energy $V = 0$, $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgrCos\theta$. Show that

$$\ddot{\theta} = -\frac{g}{l}Sin\theta.$$

(c) Suppose that a particle sliding on a cylindrical surface with Lagrangian

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgrCos\theta$$

Subject to the constraint $\psi_1 = r - l = 0$. Obtain the critical angle θ at which the particle takes off from the surface.