1. (a) Let $R$ be any ring and let $A = R \times \mathbb{Z}$ where $\mathbb{Z}$ is the set of integers. Define $+$ and $\cdot$ in $A$ by:

\[
(a, m) + (b, n) = (a + b, m + n),
\]
\[
(a, m) \cdot (b, n) = (ab + na + mb, mn), \forall a, b \in R \text{ and } m, n \in \mathbb{Z}.
\]

i. Show that $(A, +, \cdot)$ is a ring with $(0,1)$ as the unity.

ii. If $\phi : R \to A$ is a mapping defined by $\phi(a) = (a, 0) \forall a \in R$, show that $\phi$ is an injective homomorphism.

(b) i. Let $R$ be any ring with unity and for each $a \in R$, let there exist $x \in R$ such that $a^2x = a$. Show that $ax = xa$ and also show that $ax$ and $xa$ are idempotents in $Z(R)$, the center of $R$.

ii. Let $B = \{0, 2, 4, 6, 8\}$. Show that $B$ is a subring of $\mathbb{Z}_{10}$, the ring of integers modulo 10 with unity different from the unity of $\mathbb{Z}_{10}$ and state the unity of $B$.

iii. Let $R$ be a commutative ring and let $a$ and $b$ be nilpotent elements of $R$. Show that $(a + b)$ is also nilpotent.

(c) Let $R$ be a ring and let $I$ be a subset of $R$. Let

\[
r(I) = \{r \in R : Ir = 0\} \text{ and } l(I) = \{r \in R : rI = 0\}.
\]

i. Show that $r(I)$ and $l(I)$ are right and left ideals of $R$ respectively.

ii. Given that $A$ is an ideal in $R$, show that $r(I)$ and $l(I)$ are ideals in $R$.

(d) If $R = \mathbb{Z}$ and $I = (42)$, $J = (132)$ are ideals of $R$, compute $(I:J)$, the ideal quotient of $I$ and $J$.

2. (a) i. Let $I$ be an ideal of $R$ and define the multiplication map $*: [R/I] \times R \to R/I$ by

\[
*(m + I, n) = mn + I.
\]

Show that $R/I$ is a right $R$-module.

ii. Show that the direct product of two distinct $R$-modules is also an $R$-module.

iii. Let $(M_i)_{i \in I}$ be a family of $R$-submodules of an $R$-module $M$. Show that $\bigcap_{i \in I} M_i$ is also an $R$-submodule.
iv. Let $M$ be an $R$-module and for $m \in M$, let $K$ be a set defined by

$$K = \{rm + nm : r \in R, n \in \mathbb{Z}\}.$$ 

Show that $K$ is an $R$-submodule of $M$.

(b) i. Let $M$ be an $R$-module and let $r$ be some fixed element of $R$. Show that the mapping $f : M \to M$ defined by $f(m) = rm \forall m \in M$ is an $R$-homomorphism.

ii. Let $A$ and $B$ be $R$-submodules of $R$-modules $M$ and $N$ respectively. Show that

$$[M \times N]/[A \times B] \cong [M/A] \times [N/B].$$

(c) Define the following:

i. Exact sequence

ii. Short exact sequence

iii. Split exact sequence.

iv. Cokernel

v. Coimage

(d) Draw a commutative diagram of $R$-modules with exact rows and columns.

(a) i. Let $U$ and $V$ be vector spaces over the field $F$. Show that

$$\text{Hom}_F(U, V) \cong F^{m \times n}.$$ 

ii. Compute the rank of the linear mapping $\phi : \mathcal{R}^5 \to \mathcal{R}^4$ given by

$$\phi(a, b, c, d, e) = (2a + 3b + c + 4e, 3a + b + 2c - d + e, 4a - b + 3c - 2d - 2e, 5a + 4b + 3c - d + 6e).$$ 

(b) Let $A = \begin{bmatrix} -x & 4 & -2 \\ -3 & 8 - x & 3 \\ 4 & -8 & -2 - x \end{bmatrix}$ be a given matrix. Compute:

i. the invariant factors of $A$ over the ring $\mathbb{Q}[x]$,

ii. the rank of $A$.

(a) Let $\mathcal{B} = \{\sin x, \cos x, \sin 2x, \cos 2x\}$ and $V = \text{span}(B)$. In the space of all continuous functions on $\mathcal{R}$, $V$ is a four-dimensional subspace with basis $B$. Define $\phi : V \times V \to \mathcal{R}$ by

$$\phi(f, g) = f'(0) \cdot g''(0).$$ 

Show that $\phi$ is a bilinear form on $V$ and compute its matrix representation wrt the basis $B$.

(b) Let $q$ be the quadratic form associated with the symmetric bilinear form $f$. Show that:

i. $2f(u, v) = q(u + v) - q(u) - q(v)$.

ii. $4f(u, v) = q(u + v) - q(u - v)$.

(c) Reduce the quadratic polynomial

$$q(a, b, c, d) = a^2 + 2ab + 2b^2 + 6c^2 - 4ac - 10bc + 11d^2 - 6ad - 2bd + 18cd$$

to a diagonal form and state its rank and signature.