1. (i) Let \{\phi_n\} \ (n = 1, 2, 3, \ldots) be an infinite set of functions on the interval \(a \leq x \leq b\). When is \{\phi_n\} said to be an \((\alpha)\) orthogonal system, \((\beta)\) orthonormal system with respect to the weight function \(r(x)\) defined on \(a \leq x \leq b\).

(ii) Show that if the functions \(g_1(x), g_2(x), \ldots\) form an orthogonal set on \(a \leq x \leq b\), then the functions \(g_1(ct+k), g_2(ct+k), \ldots\) where \(c > 0\) form an orthogonal set on the interval \(\frac{a-k}{c} \leq t \leq \frac{b-k}{c}\).

(b) Let \(\{P_n\}\), \(n = 1, 2, 3, \ldots\) be an infinite set of polynomial functions such that

(i) \(P_n\) is of degree \(n\), \(n = 0, 1, 2, 3, \ldots\)

(ii) \(P_n(1) = 1\), \(n = 0, 1, 2, 3, \ldots\) and,

(iii) the set \(\{P_n\}\) is an orthogonal system with respect to the weight function \(r\) such that \(r(x) = 1\) on the interval \(-1 \leq x \leq 1\).

Construct consecutively the members \(P_0, P_1, P_2\) and \(P_3\) of this set by writing

\[

P_0(x) = a_0,
\]

\[
P_1(x) = b_0 x + b_1,
\]

\[
P_2(x) = c_0 x^2 + c_1 x + c_2.
\]

\[
P_3(x) = d_0 x^3 + d_1 x^2 + d_2 x + d_3,
\]

and determining the constraints in each expression so that it has the value 1 at \(x = 1\) and is orthogonal to each of the preceding expressions with respect to \(r\) on \(-1 \leq x \leq 1\).

2. a. Given the second order differential equation

\[
a(x)y'' + b(x)y' + c(x)y = f(x). \tag{2.1}
\]

Obtain the self-adjoint form

\[
(p(x)\frac{dy}{dx})' + q(x)y = F(x) \tag{2.2}
\]

from (2.1). Hence or otherwise resolve the Legendre’s differential equation

\[
(1 - x^2)y'' - 2xy + n(n+1)y = 0
\]

in the form (2.2).

b. Show that the Legendre’s polynomials, \(P_n(x)\) are orthogonal with respect to the weight function, \(w(x) = 1\) over the interval \([-1, 1]\).

3. a. Find the eigenvalues and eigenfunctions of the given boundary value problem

\[
y'' + \lambda y = 0 \quad (\lambda > 0)
\]

\[
y(0) = 0, y'(1) = 0.
\]
b. Find the Fourier series of the sawtooth function
\[ f(x) = \begin{cases} 
  x + 1, & -1 \leq x \leq 0 \\
  1 - x, & 0 \leq x \leq 1 
\end{cases} \]
\[ f(x + 2) = f(x) \]
Hence determine a series for \( \pi^2 / 8 \).

SECTION B

4. a. Define Fourier integral transform for a function \( f(x) \).
b. Solve the boundary value problem
\[ y''(x) - k^2 y(x) = -f(x), \quad -\infty < x < \infty \]
where \( k \) is a constant and \( f(x) \) is specified such that \( y(x), y'(x) \to 0 \) as \( |x| \to \infty \) using Fourier Integral transform.
c. Obtain the solution for the other form of problem in (b) written as
\[ y''(x) + k^2 y(x) = -f(x), \quad -\infty < x < \infty. \]

5. a. State without proof the convolution theorem for Fourier transforms.
b. Given the Cauchy problem for wave equation
\[ U_{tt} - c^2 U_{xx} = 0, \quad -\infty < x < \infty, t > 0 \]
where
\[ U(x, 0) = f(x), \quad -\infty < x < \infty \]
\[ U_t(x, 0) = g(x), \quad -\infty < x < \infty. \]
Show that
\[ U(\lambda, 0) = F(\lambda) \quad \text{and} \quad \frac{\partial U(\lambda, 0)}{\partial t} = G(\lambda). \]
c. Obtain the complete solution of the problem in b.

6. a. State the Fourier's Integral theorem for the function \( f(x) \) and state sufficient conditions for the theorem.
b. Give a brief statement of the Parseval's identity for the functions \( f(x) \) and \( g(x) \).
c. Given the boundary value problem
\[ U_{xx} + U_{yy} = 0, \quad 0 < x < \infty, 0 < y < \infty \]
with the boundary data
\[ U(0, y) = 0, U(x, y) \to 0 \text{ as } x \to \infty, \text{ uniformly in } y, \]
\[ U(x, \alpha) = 0, U(x, 0) = f(x). \]
Show that
\[ \frac{\partial^2 U_s(\lambda, y)}{\partial y^2} - \lambda^2 U_s(\lambda, y) = 0. \]