

MTS 441: Ordinary Differential Equations (3 Units)

PRQT MTS 342

Some Course Content

Existence and Uniqueness theorems, dependence of solutions on initial data and parameter. Properties of solutions. Sturm comparison theorem. Linear system; Floquet's theorem, Nonlinear system, Stability theory.

1.0 Initial Value Problem

Let $\mathbb{R} = (-\infty, \infty)$ and $D \subset \mathbb{R}^2$ be an open, connected subset of \mathbb{R}^2 . Let $f \in C'(D, \mathbb{R})$ and $x' = \frac{dx}{dt}$.

$$x' = f(t, x) \tag{1.1}$$

is an ordinary differential equation of the first order.

Definition

A solution of (1.1) on an open interval $I = (a, b)$ is a real-valued continuously differentiable function $\phi(t)$ such that

- (i) $(t, \phi(t)) \in D$ for all $t \in I$ and
- (ii) $\phi'(t) = f(t, \phi(t))$ for all $t \in I$.

Suppose in addition to (1.1) we have a condition $x(\tau) = \xi$ i.e.

$$\begin{aligned} x' &= f(t, x) \\ x(\tau) &= \xi \end{aligned} \tag{1.2}$$

then (1.2) is called an initial value problem. Thus, a function ϕ is a solution of (1.2) if ϕ is a solution of (1.1) on I which contains τ and $\phi(\tau) = \xi$.

(1.2) can be represented equivalently by an integral equation. Let ϕ be a solution of (1.2), it follows that

$$\begin{aligned} \phi'(t) &= f(t, \phi(t)) \\ \phi(\tau) &= \xi \\ \frac{d\phi}{dt} &= f(t, \phi) \\ d\phi &= f(t, \phi)dt \end{aligned}$$

$$\int_{\tau}^t d\phi = \int_{\tau}^t f(s, \phi(s)) ds$$

$$\phi(s)|_{\tau}^t = \int_{\tau}^t f(s, \phi(s)) ds$$

$$\phi(t) = \phi(\tau) + \int_{\tau}^t f(s, \phi(s)) ds$$

Hence

$$\phi(t) = \xi + \int_{\tau}^t f(s, \phi(s)) ds \quad (1.3)$$

required integral equation.

Remark: (1.3) \Rightarrow (1.2). Find out.

1.2 Systems of first order ordinary differential equations

A system of n first-order ordinary differential equation for n unknown functions $x_1(t), \dots, x_n(t)$ is of the form

$$\begin{aligned} x'_1 &= f_1(t, x_1, \dots, x_n) \\ x'_2 &= f_2(t, x_1, \dots, x_n) \\ &\cdot \\ &\cdot \\ x'_n &= f_n(t, x_1, \dots, x_n) \end{aligned} \quad (1.4)$$

or equivalently,

$$x'_i = f_i(t, x_1, \dots, x_n) \quad (i = 1, 2, \dots, n) \quad (1.5)$$

where each f_i is a function of $n + 1$ variables t, x_1, \dots, x_n . The system (1.4) can be written in a vector form

$$x' = f(t, x) \quad (1.6)$$

Where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

$$x' = \begin{pmatrix} x'_1 \\ x'_2 \\ \cdot \\ \cdot \\ x'_n \end{pmatrix}$$

and

$$f(t, x) = \begin{pmatrix} f_1(t, x) \\ f_2(t, x) \\ \cdot \\ \cdot \\ f_n(t, x) \end{pmatrix}$$

By a solution of (1.6), we mean a real-valued continuously differentiable function $(\phi_1, \phi_2, \dots, \phi_n)$ defined on I such that $(t, \phi_1(t), \phi_2(t), \dots, \phi_n(t)) \in D \subset \mathbb{R}^{n+1}$ for all $t \in I$ and $\phi'_i = f_i(t, \phi_1(t), \phi_2(t), \dots, \phi_n(t))$ for all $t \in I$. $i = 1, 2, \dots, n$.

The initial value problem associated with (1.6) is

$$x' = f(t, x) \tag{1.7}$$

$$x(\tau) = \xi$$

Remark: Equation (1.7) can be represented thus

$$\phi(t) = \xi + \int_{\tau}^t f(s, \phi(s)) ds \tag{1.8}$$

where

$$\begin{pmatrix} \phi_1(t) \\ \phi_2(t) \\ \cdot \\ \cdot \\ \phi_n(t) \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \cdot \\ \cdot \\ \xi_n \end{pmatrix} + \begin{pmatrix} \int_{\tau}^t f_1(s, \phi(s)) ds \\ \int_{\tau}^t f_2(s, \phi(s)) ds \\ \cdot \\ \cdot \\ \int_{\tau}^t f_n(s, \phi(s)) ds \end{pmatrix}$$

Definition: If, in (1.6), $f = f(x)$ is explicitly independent of t , then (1.6) is said to be autonomous, otherwise it is said to be non-autonomous.

Any higher-order ordinary differential equation is reducible to an equivalent first-order system (1.6). To illustrate the procedure consider the single n th- order ordinary differential equation for the unknown function $y(t)$:

$$y^{(n)} = h(t, y, y', \dots, y^{(n-1)}) \quad (1.9)$$

where h is a specified function of $t, y, y', \dots, y^{(n-1)}$

Define

$$x_1 = y, x_2 = y', \dots, x_n = y^{(n-1)} \quad (1.10)$$

and form the equivalent system

$$\begin{aligned} x'_1 &= x_2 \\ x'_2 &= x_3 \\ &\cdot \\ &\cdot \\ x'_{n-1} &= x_n \\ x'_n &= h(t, x_1, x_2, \dots, x_n) \end{aligned} \quad (1.11)$$

Comparing (1.11) and (1.6) we have that

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

$$f(t, x) = \begin{pmatrix} x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_n \\ h(t, x_1, x_2, \dots, x_n) \end{pmatrix}$$

We conclude that any results obtained for the first-order system (1.6) have their counterparts for the n th-order equation (1.9). To complete the correspondence, we note that corresponding to the initial condition in (1.7), the transformation (1.11) determines the appropriate initial conditions for (1.9), which are

$$y(\tau) = y_1, y'(\tau) = y_2, \dots, y^{(n-1)}(\tau) = y_n \quad (1.12)$$

Then ξ in (1.7) is the vector with components y_1, y_2, \dots, y_n .

1.3 Existence and uniqueness theorem for system of first order ordinary differential equations

We shall be concerned with normal systems of ordinary differential equations of the form

$$\dot{x} = f(t, x) \quad (1.13)$$

where $x: I(\subset \mathbb{R}) \rightarrow \mathbb{R}^n$, $f: I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$

and the dot denotes differentiation with respect to the independent variable t .

The initial value problem associated with (1.13) is

$$\begin{aligned} \dot{x} &= f(t, x) \\ x(t_0) &= x_0 \end{aligned} \quad (1.14)$$

The initial value problem consists of finding a solution $x(t, t_0, x_0)$ for given $t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n$ which reduces to x_0 at $t = t_0$ that is

$$x(t_0, t, x_0) = x_0.$$

Theorem 1.1

Let $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ be defined and continuous in a certain domain $D \subset \mathbb{R}^{n+1}$ and suppose that for any two points $(t, x), (t, y) \in D$

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\| \tag{1.15}$$

where $L > 0$ is a constant (which may depend on D and on f).

Then for every point $(t_0, x_0) \in D$ there exists a solution $x = \phi(t)$ of (1.14) that is defined in some interval containing t_0 and which satisfies $\phi(t_0) = x_0$. Furthermore if there exists two solutions $x_1 = \phi(t)$ and $x_2 = \phi(t)$ both satisfying (1.14) and each solution defined on some interval containing t_0 then the two solutions coincide whenever both are defined.

(1.15) is called Lipschitz condition and L the Lipschitz constant.

2.0 LINEAR SYSTEMS

2.1 Uniqueness and Existence Theorems for a Linear System

In this chapter we study linear differential equations which are a very special but very important class of differential equations. That is, we study systems of differential equations of the form

$$\begin{aligned}
 x'_1 &= a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + h_1(t) \\
 x'_2 &= a_{21}(t)x_1 + \cdots + a_{2n}(t)x_n + h_2(t) \\
 &\cdots \\
 &\cdots \\
 &\cdots \\
 x'_n &= a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + h_n(t)
 \end{aligned}
 \tag{2.1.1}$$

in which the right-hand sides of the equations are linear in x_1, \dots, x_n .

The study of such systems is very important for the following reasons:

- (i) Equations of this form often arise in problems in physics and engineering;
- (ii) From the view point of pure mathematics, the study is important because an elegant and complete theory is obtained; and
- (iii) As obtained in other parts of linear and nonlinear analysis, the theory for linear equations is the basis for much study of non-linear equations.

The component form of (2.1.1) is

$$x' = A(t)x + h(t)
 \tag{2.1.2}$$

where $A(t)$ is an $n \times n$ matrix function of t of the form

$$A(t) = \begin{pmatrix}
 a_{11}(t) & \cdots & a_{1n}(t) \\
 a_{21}(t) & \cdots & a_{2n}(t) \\
 \cdots & \cdots & \cdots \\
 \cdots & \cdots & \cdots \\
 \cdots & \cdots & \cdots \\
 a_{n1}(t) & \cdots & a_{nn}(t)
 \end{pmatrix}$$

and $h(t)$ is the vector function of t of the form

$$h(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \\ \dots \\ \dots \\ \dots \\ h_n(t) \end{pmatrix}$$

Properties of matrix $A(t)$ are very important in the study of solutions of (2.1.1). To this end, we introduce some definitions used in the study of matrix $A(t)$. Some of these definitions are found in some basic linear algebra texts.

Definition: If $A = (a_{ij})$ is a constant matrix, the norm of A , denoted $|A|$, is

$$|A| = \sum_{i,j=1}^n |a_{ij}|$$

Let A and B be constant matrices, and x a constant vector, thus

$$|A + B| \leq |A| + |B|$$

$$|AB| \leq |A||B|$$

$$|Ax| \leq |A||x|$$

Definition: Let $A(t) = (a_{ij}(t))$, then the derivative of $A(t)$, sometimes denoted $\frac{dA(t)}{dt}$ is

$\frac{d}{dt} a_{ij}(t)$, the integral of $A(t)$ over $[a, b]$, sometimes denoted by $\int_a^b A(t) dt$ is $(\int_a^b a_{ij}(t) dt)$, and

the trace of A , sometimes denoted by $tr A(t)$, is $\sum_{i=1}^n a_{ii}(t)$.

Existence Theorem 2.1 for Linear Systems

If for $i, j = 1, 2, \dots, n$, each $a_{ij}(t)$ is continuous for all real t and if $h(t)$ is continuous for all real t , then if (t_0, x_0) is an arbitrary point in (t, x) -space, there is a unique solution $x(t, t_0, x_0)$ of (2.1.1) such that $x(t_0, t_0, x_0) = x_0$ and solution $x(t, t_0, x_0)$ has for its domain the real t -axis.

Remark: The theorem above shows that there is no extension problem for solutions of linear equations. That is, if the elements of $A(t)$ and $h(t)$ are continuous for all t , then the solution has for its domain the entire t -axis.

Instead of proving this theorem directly, we prove a somewhat more general theorem which is used less frequently but is of sufficient interest to be presented for its own sake.

Existence Theorem 2.2 for Linear Systems

Suppose $A(t)$ and $h(t)$ are Riemann integrable functions of t on (a, b) , i.e. the Riemann integrals over any interval $[c, d]$ contained in (a, b) of the elements of $A(t)$ and $h(t)$ exist, and suppose there exists a function $k(t)$ with domain (a, b) such that

- (1) $k(t)$ is continuous and bounded on (a, b)
- (2) if $t \in (a, b)$, then $|A(t)| \leq k(t)$ and $|h(t)| \leq k(t)$.

Let $t_0 \in (a, b)$ and suppose x_0 is a fixed vector. Then equation (1.1) has a unique solution $x(t)$ on (a, b) such that $x(t_0) = x_0$ in the following sense: if $t \in (a, b)$, then

$$x(t) = x_0 + \int_{t_0}^t A(s)x(s)ds + \int_{t_0}^t h(s)ds \quad (2.1.3)$$

Proof

For $t \in (a, b)$, we define

$$\begin{aligned}x_0(t) &= x_0 \\&\vdots \\&\vdots \\x_{n+1}(t) &= x_0 + \int_{t_0}^t A(s)x_n(s)ds + \int_{t_0}^t h(s)ds \quad (n = 0, 1, 2, \dots)\end{aligned}$$

If $x_n(t)$ is continuous on (a, b) , then $A(s)x_n(s)$ is integrable over any interval $[c, d]$ contained in (a, b) and hence $x_{n+1}(t)$ is defined and continuous on (a, b) . To show that the $x_n(t)$ converge uniformly, we proceed as follows. First if $t \in (a, b)$,

$$\begin{aligned}|x_1(t) - x_0(t)| &\leq \int_{t_0}^t \{|A(s)|x(s)| + |h(s)|\}ds \\&\leq (1 + |x_0|) \int_{t_0}^t k(s)ds\end{aligned}$$

Let $\int_{t_0}^t k(s)ds$ and assume that for $t \in (a, b)$

$$|x_n(t) - x_{n-1}(t)| \leq (1 + |x_0|) \frac{(k(t))^n}{n!}$$

Then

$$\begin{aligned}|x_{n+1}(t) - x_n(t)| &\leq \int_{t_0}^t |A(s)x_n(s) - A(s)x_{n-1}(s)|ds \\&\leq (1 + |x_0|) \int_{t_0}^t k(s) \frac{(k(s))^n}{n!} ds\end{aligned}$$

Since

$\frac{d}{dt} K(t) = k(t)$ and $K(t_0) = 0$, then

$$|x_{n+1}(t) - x_n(t)| \leq (1 + |x_0|) \frac{(k(t))^{n+1}}{(n+1)!}$$

Thus, the sequence $\{x_n(t)\}$ converges uniformly on any closed interval $[c, d]$ in (a, b) to a continuous function $x(t)$. To complete the proof of the existence of the solution, it is sufficient to show that

$$\lim_{n \rightarrow \infty} \int_{t_0}^t A(s)x_n(s)ds = \int_{t_0}^t A(s)x(s)ds$$

Since $x(t)$ is continuous, $\int_{t_0}^t A(s)x(s)ds$ exists and

$$\left| \int_{t_0}^t A(s)[x_n(s) - x(s)]ds \right| \leq \int_{t_0}^t |A(s)||x_n(s) - x(s)|ds \leq \varepsilon M$$

Where M is a bound for $k(t)$ on (a, b) .

The proof that $x(t)$ is a unique solution in any closed interval $[c, d]$ in (a, b) is given below.

Suppose there exists solutions $x(t)$ and $y(t)$ of (2.1.3) on an interval $(t_0 - r, t_0 + r)$, where r is a positive number, such that $x(t_0) = y(t_0) = x_0$. By induction, we obtain an estimate on $|x(t) - y(t)|$ for $t \in [t_0, t_0 + r - \delta]$ where $0 < \delta < r$. A similar estimate can be obtained for $t \in [t_0 - r + \delta, t_0]$. Since $x(t), y(t)$ are continuous on $[t_0, t_0 + r - \delta]$ for fixed δ there exists $B > 0$ such that, if $t \in [t_0, t_0 + r - \delta]$, then $|x(t) - y(t)| \leq B$. But

$$\begin{aligned} |x(t) - y(t)| &\leq \int_{t_0}^t |A(s)x(s) - A(s)y(s)|ds \\ &\leq k \int_{t_0}^t |x(s) - y(s)|ds \end{aligned} \tag{2.1.4}$$

Therefore

$$|x(t) - y(t)| \leq kB(t - t_0)$$

Assume that $|x(t) - y(t)| \leq \frac{k^m}{m!} B(t - t_0)^m$, for m a positive integer. Then by (2.1.4),

$|x(t) - y(t)| \leq \frac{k^{m+1}}{(m+1)!} B(t - t_0)^{m+1}$, which is the $(m + 2)$ th term in the (convergent) series for

$Be^{k(t-t_0)}$. Therefore $|x(t) - y(t)| < \varepsilon$; hence $x(t) = y(t)$ for $t \in [t_0, t_0 + r - \delta]$. Since δ is arbitrarily small, $x(t) = y(t)$ for $t \in (t_0, t_0 + r)$.

The proof of Existence Theorem 2.1 for Linear Systems is obtained from Existence Theorem 2.2 for Linear Systems thus. If the elements of $A(t)$ and $h(t)$ are continuous then since the solution $x(t)$ is continuous, equation (2.1.3) may be differentiated with respect to t and we obtain:

$$\frac{dx(t)}{dt} = A(t)x(t) + h(t).$$

2.2 Homogenous Linear Systems

First-order system of differential equations of the form

$$\begin{aligned} x'_1 &= a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n \\ x'_2 &= a_{21}(t)x_1 + \cdots + a_{2n}(t)x_n \\ &\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ &\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ &\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ x'_n &= a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n \end{aligned} \tag{2.2.1}$$

in which the right hand sides of the equations are linear in x_1, \dots, x_n , is called homogenous. In compact form, this is written as

$$x' = A(t)x \tag{2.2.2}$$

where $A(t)$ is an $n \times n$ matrix function of t , continuous for $t \in [a, b]$.

Equivalently, first-order system (2.1.2) is said to be homogenous whenever $h(t)$ is identically zero.

The homogenous system (2.2.2) has two important properties:

- (i) The identically zero function, $x(t) = 0$ for all $t \in [a, b]$ is a solution of (2.2.2), and is the unique solution such that $x(t_0) = 0$ for any $t \in [a, b]$;

- (ii) If $x_1(t)$ and $x_2(t)$ are solutions of (2.2.2), then so is the linear combination of $x_1(t)$ and $x_2(t)$ i.e. $x(t) = a_1x_1(t) + a_2x_2(t)$ for any two scalar constants a_1 and a_2 .

Thus, the set of all solutions of (2.2.2) (equivalently (2.2.1)) on an interval $[a, b]$ form a vector space. The following definitions will assist us determine the dimension of this space.

Definition: A set of n functions $x_1(t), \dots, x_n(t)$ are linearly dependent on $[a, b]$ if there exist scalar constants a_1, \dots, a_n , not all zero, such that

$$\sum_{i=1}^n a_i x_i = 0 \text{ (for } t \in [a, b]).$$

Otherwise they are linearly independent.

For example, with $n = 2$ the vectors

$$\begin{pmatrix} 1 \\ t^2 \end{pmatrix}, \begin{pmatrix} t \\ 1 \end{pmatrix}$$

are linearly independent vector functions of t for t in any interval $[a, b]$, but at $t = 1$ both reduce to

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and, hence are linearly dependent 2-vectors for this particular value of t .

Theorem 2.3

The set of all solutions of (2.2.2) on an interval $[a, b]$ form an n -dimensional vector space.

Proof

It was earlier shown that the set of all solutions form a vector space. Next we show that there exist n linearly independent solutions. Let d_1, \dots, d_n be n linearly independent n -vectors and, using theorem 2.1, let $x_r(t)$ be the unique solution of (2.2.2) such that

$$x_r(t_0) = d_r \quad (r = 1, 2, \dots, n),$$

where $t_0 \in I$. Then $x_1(t), \dots, x_n(t)$ are linearly independent on $[a, b]$. For suppose that the contrary holds and there exist constants a_1, \dots, a_n such that

$$a_1 x_1(t) + \dots + a_n x_n(t) = 0 \quad (\text{for all } t \in [a, b]).$$

Then, putting $t = t_0$, it follows that

$$a_1 d_1 + \dots + a_n d_n = 0$$

But, since d_1, \dots, d_n are linearly independent n -vectors, it follows that $a_1 = 0, \dots, a_n = 0$. This establishes that the dimension of the vector space is at least n .

To show that the dimension is exactly n , we next demonstrate that any solution $x(t)$ of (2.2.2) can be written as a linear combination of $x_1(t), \dots, x_n(t)$.

Indeed, given $x(t)$, let $x_0 = x(t_0)$, where $t_0 \in [a, b]$. Then, since d_1, \dots, d_n are linearly independent n -vectors, there exists a unique set of constants a_1, \dots, a_n such that

$$x_0 = a_1 d_1 + \dots + a_n d_n.$$

Now consider

$$a_1 x_1(t) + \dots + a_n x_n(t)$$

By construction, this is a solution of (2.2.2) and satisfies the initial condition $x(t_0) = x_0$. But theorem 2.1 states that there is a unique solution to the initial-value problem and, hence, it must be identically equal to $x(t)$. We have shown that

$$x(t) = a_1 x_1(t) + \dots + a_n x_n(t)$$

for all $t \in [a, b]$. Consequently, theorem 2.1 allows us to transfer linear independence of the initial conditions at t_0 to the solution for all $t_0 \in [a, b]$.

Theorem 2.3 shows that the general solution of (2.2.2) is

$$x(t) = a_1 x_1(t) + \dots + a_n x_n(t) \quad (2.2.3)$$

where $x_1(t), \dots, x_n(t)$ are n linearly independent solutions and a_1, \dots, a_n are n arbitrary constants.

Definition: Let $x_1(t), \dots, x_n(t)$ be n solutions of (2.2.2) on an interval $[a, b]$, and put

$$X(t) = (x_1(t), \dots, x_n(t)) \quad (2.2.4)$$

where $X(t)$ is an $n \times n$ matrix solution of

$$X' = AX \quad (2.2.5)$$

X is called a fundamental matrix if $x_1(t), \dots, x_n(t)$ are linearly independent. If, in addition, $X(t_0) = E$, the unit matrix, then $X(t)$ is the principal fundamental matrix. Further

$$W(t) = \det X(t) \quad (2.2.6)$$

is called the Wronskian.

The property (2.2.5) is immediate from the definition (2.2.4) and (2.2.2). Further, if $X(t)$ is a fundamental matrix solution of (2.2.5), then so is $X(t)C$ for any non-singular constant matrix C .

Indeed, let

$$Y(t) = X(t)C$$

Then $Y(t)$ is non-singular, and

$$Y' = X' C = AXC = AY$$

Note that the column of Y are linear combinations of the columns of X . Also the general solution (2.2.3) can be written in the form

$$x(t) = X(t)c$$

where c is an arbitrary n -vector, with components c_1, \dots, c_n

2.3 Solution of Autonomous Systems with Constant Coefficients

To find the general solution of the system of differential equations

$$X' = MX \quad (2.3.1)$$

where $X' = \frac{dx}{dt}$, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \end{pmatrix}$,

assume a solution of the form

$$X(t) = \begin{pmatrix} A \\ B \end{pmatrix} e^{rt} \quad (2.3.2)$$

Substitute (2.3.2) into (2.3.1) and rearrange the result to obtain

$$\begin{pmatrix} a-r & b \\ c & d-r \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.3.3)$$

Solve the characteristic equation

$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = 0$$

for r , obtaining the roots (eigenvalues) r_1 and r_2 .

Next, we search for a fundamental matrix for this system of equations. This depends on the nature of the eigenvalues: whether they are distinct – either real or complex – or repeated.

- (i) For real, distinct eigenvalues r_1 and r_2 , substitute one of the eigenvalues – say, r_1 – into (2.3.3) and solve the resulting algebraic equations for the associated eigenvector. Repeat this process for the other eigenvalue r_2 . The columns in a fundamental matrix are formed by these eigenvectors times e^{rt} , for the appropriate value of r .
- (ii) For a repeated eigenvalue r , substitute

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} e^{rt} + \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} t e^{rt} \quad (2.3.4)$$

into (2.3.3) and solve the resulting algebraic equations for A_1, B_1, A_2 and B_2 . The result of this operation will contain two arbitrary constants. The columns of a

fundamental matrix in this case are composed of the vector functions that multiply the two arbitrary constants.

- (iii) For complex eigenvalues r_1 and r_2 , substitute one of them – say, $r_1 = \alpha + i\beta$ – into (2.3.3) and solve the resulting algebraic equations. This gives a complex eigenvector, which when multiplied by $e^{(\alpha+i\beta)t}$ will result in a complex valued solution. Form a fundamental matrix by having its columns as the real and imaginary parts of this solution.

For each of these cases mentioned above, the general solution of the system of differential equations is

$$X(t) = UC,$$

Where U is a fundamental matrix and C is a vector of arbitrary constants, $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

If in addition to (2.3.1) is the initial value $x(0) = x_0, y(0) = y_0$ - evaluate the arbitrary constants by solving the system

$$U(0)C = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

for c_1 and c_2 .

Example: Solve the system of equations

$$x' = 2x + 5y$$

$$y' = x + 6y$$

Solution: We can rewrite the system of equations above in matrix form

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 5 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.3.5}$$

It follows that in this case

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \text{ and } M = \begin{pmatrix} 2 & 5 \\ 1 & 6 \end{pmatrix}$$

We assume a nontrivial solution

$$X(t) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{rt} \quad (2.3.6)$$

where A, B are arbitrary constants. The derivative of $X(t)$ with respect to t , that is,

$$X'(t) = \frac{dx}{dt} = \frac{d}{dt} \begin{pmatrix} A \\ B \end{pmatrix} e^{rt} = \begin{pmatrix} rA \\ rB \end{pmatrix} e^{rt} \quad (2.3.7)$$

If we substitute (2.3.6) and (2.3.7) into (2.3.5), we obtain

$$\begin{pmatrix} rA \\ rB \end{pmatrix} e^{rt} = \begin{pmatrix} 2 & 5 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} e^{rt}$$

Because the common factor e^{rt} is never zero, we may divide by it and rearrange the preceding equation to obtain

$$\begin{pmatrix} 2 & 5 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} - \begin{pmatrix} rA \\ rB \end{pmatrix} = \begin{pmatrix} 2-r & 5 \\ 1 & 6-r \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This system of algebraic equations will have a nontrivial solution only if the determinant of the coefficients is zero, so

$$\begin{vmatrix} 2-r & 5 \\ 1 & 6-r \end{vmatrix} = 0$$

This gives the quadratic equation

$$(2-r)(6-r) - 5 = r^2 - 8r + 7 = 0$$

Or

$$(r-1)(r-7) = 0$$

The values $r = 1$ and $r = 7$ are the only values of r that give nontrivial solutions of the system of equations in (2.3.8). If we substitute $r = 1$ in (2.3.8) we obtain

$$\begin{pmatrix} 1 & 5 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

So our only condition is that $B = -\frac{1}{5}A$.

This gives the vector

$$\begin{pmatrix} A \\ 1 \\ -\frac{1}{5}A \end{pmatrix},$$

where A is an arbitrary constant, and the solution

$$\begin{pmatrix} A \\ 1 \\ -\frac{1}{5}A \end{pmatrix} e^t = A \begin{pmatrix} 1 \\ 1 \\ -\frac{1}{5} \end{pmatrix} e^t = A \begin{pmatrix} e^t \\ 1 \\ -\frac{1}{5}e^t \end{pmatrix}$$

If we use $r = 7$ in (2.3.8) we obtain

$$\begin{pmatrix} -5 & 5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

So our only condition is that $B = A$. This gives the vector

$$\begin{pmatrix} A \\ A \end{pmatrix},$$

Where A is an arbitrary constant, and the solution

$$\begin{pmatrix} A \\ A \end{pmatrix} e^{7t} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t} = A \begin{pmatrix} e^{7t} \\ e^{7t} \end{pmatrix}$$

In these two solutions we have two different arbitrary constants, each denoted by the symbol A .

If we designate these arbitrary constants C_1 and C_2 , we can express our explicit solution of (2.3.5) as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -\frac{1}{5} \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t}$$

Using the fact that

$$C_1 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + C_2 \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

we may also write this solution in matrix form as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^t & e^{7t} \\ -\frac{1}{5}e^t & e^{7t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Or

$$X(t) = UC$$

where

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, U = \begin{pmatrix} e^t & e^{7t} \\ -\frac{1}{5}e^t & e^{7t} \end{pmatrix}, \text{ and } C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Remark

1. The two solutions of the characteristic equation, in this case $r = 1$ and $r = 7$ are called eigenvalues. The two vectors of constants associated with these eigenvalues, in this case

$$A \begin{pmatrix} 1 \\ -\frac{1}{5} \end{pmatrix} \text{ and } A \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

are called eigenvectors.

The example we just treated has real distinct eigenvalues.

Example: Real, Repeated Eigenvalue

Consider the system of differential equations

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 9 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (2.3.9)$$

where we seek a solution of the form

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{rt}$$

Substituting this expression into (2.3.9) yields

$$\begin{pmatrix} rA \\ rB \end{pmatrix} e^{rt} = \begin{pmatrix} 2 & 9 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} e^{rt}$$

and the associated set of algebraic equations

$$\begin{pmatrix} 2-r & 9 \\ -1 & -4-r \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.3.10)$$

From the determinant of the preceding 2×2 matrix we can obtain the characteristic equation as

$$(2 - r)(-4 - r) + 9 = r^2 + 2r + 1 = (r + 1)^2 = 0$$

giving the repeated eigenvalue of $r = -1$.

To find the eigenvector associated with the eigenvalue $r = -1$, we substitute $r = -1$ into (2.3.10) and obtain

$$\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This requires that $A = -3B$, so we have the eigenvector

$$\begin{pmatrix} -3B \\ B \end{pmatrix}$$

Thus, one solution of (2.3.9) is

$$\begin{pmatrix} -3B \\ B \end{pmatrix} e^{-t}$$

The other solution is given by

$$\begin{pmatrix} -3B \\ B \end{pmatrix} t e^{-t}$$

Then the explicit solution of (2.3.9) is taken as

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -3A_1 \\ B_1 \end{pmatrix} e^{-t} + \begin{pmatrix} -3A_2 \\ B_2 \end{pmatrix} t e^{-t} \quad (2.3.11)$$

Substituting this expression into (2.3.9) gives

$$\begin{pmatrix} 3A_1 - 3A_2 \\ B_2 - B_1 \end{pmatrix} e^{-t} + \begin{pmatrix} 3A_2 \\ -B_2 \end{pmatrix} t e^{-t} = \begin{pmatrix} -6A_1 + 9B_1 \\ 3A_1 - 4B_1 \end{pmatrix} e^{-t} + \begin{pmatrix} -6A_2 + 9B_2 \\ 3A_2 - 4B_2 \end{pmatrix} t e^{-t}$$

Equating coefficients of e^{-t} and $t e^{-t}$ gives the system of algebraic equations

$$3A_1 - 3A_2 = -6A_1 + 9B_1$$

$$B_2 - B_1 = 3A_1 - 4B_1$$

$$3A_2 = -6A_2 + 9B_2$$

$$-B_2 = 3A_2 - 4B_2$$

This system of equations has the solution

$$A_1 = B_1 + \frac{1}{3}B_2$$

$$A_2 = B_2$$

where B_1 and B_2 may be chosen arbitrarily, so we denote them by C_1 and C_2 . This means that the solution (2.3.11) may be written in the form

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -(3C_1 + C_2) \\ C_1 \end{pmatrix} e^{-t} + \begin{pmatrix} -3C_2 \\ C_2 \end{pmatrix} t e^{-t},$$

Or

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -3e^{-t} & -(1+3t)e^{-t} \\ e^{-t} & t e^{-t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Here a fundamental matrix is

$$U = \begin{pmatrix} e^{-t} & -(1+3t)e^{-t} \\ e^{-t} & t e^{-t} \end{pmatrix}$$

Example: Complex Eigenvalues

Consider the system of differential equations

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (2.3.12)$$

In a manner similar to the previous example we seek a solution of the form

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{rt}$$

and determine the characteristic equation from the determinant of the 2×2 matrix in the equation

$$\begin{pmatrix} 2-r & -1 \\ 1 & 2-r \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.3.13)$$

as

$$(2-r)(2-r) + 1 = r^2 - 4r + 5 = 0$$

The eigenvalues are $r = 2 \pm i$.

If we substitute $r = 2 \pm i$ into (2.3.13) we obtain

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so A and B must be related by $A = Bi$. This gives an eigenvector as

$$B \begin{pmatrix} i \\ 1 \end{pmatrix}$$

and a solution of (2.3.12) as

$$\begin{aligned} B \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(2+i)t} &= B \begin{pmatrix} i \\ 1 \end{pmatrix} e^{2t} e^{it} \\ &= B \begin{pmatrix} i \\ 1 \end{pmatrix} e^{2t} e^{it} (\cos t + i \sin t) \end{aligned}$$

If we decompose this vector into its real and imaginary parts we obtain

$$B \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(2+i)t} = B \begin{pmatrix} -e^{2t} \sin t \\ e^{2t} \cos t \end{pmatrix} + iB \begin{pmatrix} e^{2t} \cos t \\ e^{2t} \sin t \end{pmatrix} \quad (2.3.14)$$

as our solution. However, because our original differential equation (2.3.12) has only real coefficients and our solution contains both a real part and an imaginary part, each of these parts, namely,

$$\begin{pmatrix} -e^{2t} \sin t \\ e^{2t} \cos t \end{pmatrix} \text{ and } \begin{pmatrix} e^{2t} \cos t \\ e^{2t} \sin t \end{pmatrix}$$

must separately satisfy (2.3.12). Thus, our fundamental matrix is

$$U = \begin{pmatrix} -e^{2t} \sin t & e^{2t} \cos t \\ e^{2t} \cos t & e^{2t} \sin t \end{pmatrix}$$

Hence, our solution is

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -e^{2t} \sin t & e^{2t} \cos t \\ e^{2t} \cos t & e^{2t} \sin t \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

3.0 Sturm Theory

3.1 Self-Adjoint Equations of the Second Order

In this section we shall consider some basic properties of some second-order self-adjoint ordinary differential equations. We begin by introducing the adjoint of a second-order homogenous linear differential equation.

Definition 3.1

Consider the second-order homogenous differential equation

$$p(x) \frac{d^2y}{dx^2} + q(x) \frac{dy}{dx} + r(x)y = 0 \quad (3.1)$$

where p is twice continuously differentiable, q is once continuously differentiable, r is continuous, and $p(x) \neq 0$ on $I = [a, b]$. The adjoint equation to (3.1) is

$$\frac{d^2}{dx^2} (p(x)y) - \frac{d}{dx} (q(x)y) + r(x)y = 0,$$

that is, after taking the indicated derivatives,

$$p(x) \frac{d^2y}{dx^2} + [2p'(x) - q(x)] \frac{dy}{dx} + [p''(x) - q'(x) + r(x)]y = 0 \quad (3.2)$$

where the primes denote differentiation with respect to t .

Example 3.1. Consider

$$x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + 3y = 0$$

Here $p(x) = x^2$, $q(x) = 3x$ and $r(x) = 3$. By (3.2), the adjoint equation to this equation is

$$x^2 \frac{d^2y}{dx^2} + [4x - 3x] \frac{dy}{dx} + [2 - 3 + 3]y = 0$$

Or simply

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 2y = 0.$$

Remark: The adjoint equation of (3.1) is always the original equation (3.1) itself.

We can now consider the special case in which the adjoint equation (3.2) of (3.1) is also (3.1) itself.

Definition 3.2

The second-order homogenous linear differential equation

$$p(x) \frac{d^2 y}{dx^2} + q(x) \frac{dy}{dx} + r(x)y = 0$$

is called self adjoint if it is identical with its adjoint equation (3.2).

Theorem 3.1

Consider the second-order linear differential equation (3.1). It is called self adjoint if

$$\frac{d}{dx}(p(x)) = q(x) \quad \text{on } a \leq t \leq b.$$

Corollary 3.1

Let (3.1) be self adjoint, then

$$\frac{d}{dx} \left[p(x) \frac{dy}{dx} \right] + r(x)y = 0.$$

Example 3.2. Consider the linear differential equation

$$x^3 \frac{d^2 y}{dx^2} + 3x^2 \frac{dy}{dx} + y = 0.$$

The equation is self adjoint since $p(x) = x^3$, $q(x) = 3x^2$ and $r(x) = 1$. Thus this equation is written as

$$\frac{d}{dx} \left(x^3 \frac{dy}{dx} \right) + y = 0$$

Theorem 3.2

Given the linear differential equation

$$p(x) \frac{d^2 y}{dx^2} + q(x) \frac{dy}{dx} + r(x)y = 0,$$

where p, q and r are continuous, $p(x) \neq 0$ on I , then the equation can be transformed into the equivalent self-adjoint form

$$\frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + Q(x)y = 0,$$

where $P(x) = e^{\int \frac{q(x)}{p(x)} dx}$, $Q(x) = \frac{r(x)}{p(x)} e^{\int \frac{q(x)}{p(x)} dx}$

by multiplication throughout by the factor $\frac{1}{p(x)} e^{\int \frac{q(x)}{p(x)} dx}$.

Example 3.3

Consider the equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0.$$

Here $p(x) = x^2$, $q(x) = x$, $r(x) = 1$. Since $p'(x) = 2x \neq 1 = q'(x)$, the equation above is not self-adjoint. The factor for this equation is

$$\frac{1}{p(x)} e^{\int \frac{q(x)}{p(x)} dx} = \frac{1}{x^2} e^{\int \frac{x}{x^2} dx} = \frac{x}{x^2} = \frac{1}{x}$$

Multiplying the equation by $\frac{1}{x}$ on an interval $I = [a, b]$ which does not include $x = 0$, we obtain

$$x \frac{d^2 y}{dx^2} + \frac{dy}{dx} + \frac{1}{x} y = 0.$$

Since $\frac{d}{dx}(x) = 1$,

this equation is self-adjoint and may be written in the form

$$\frac{d}{dx} \left(x \frac{dy}{dx} \right) + \frac{1}{x} y = 0.$$

3.2 Some Basic Results of Sturm Theory

Consider the self-adjoint second-order equation in the form

$$\frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + Q(x)y = 0 \tag{3.3}$$

where P has continuous derivative, Q is continuous and $P(x) > 0$ on $I = [a, b]$.

Theorem 3.3

Let $\phi \in C'(I)$ be a solution of

$$\frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + Q(x)y = 0$$

and that ϕ has an infinite number of zeros on $[a, b]$, then $\phi(x) = 0$ for all $x \in [a, b]$.

Theorem 3.4 (Abel's Formula)

Let ϕ and θ be any two solutions of

$$\frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + Q(x)y = 0$$

on I , then for all $x \in I$,

$$P(x)[\phi(x) \theta'(x) - \phi'(x) \theta(x)] = k$$

where k is a constant.

Theorem 3.5

Let ϕ and θ be two solutions of

$$\frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + Q(x)y = 0$$

such that ϕ and θ have common zero on I , then ϕ and θ are linearly dependent on I . If in addition $\phi(x_0) = 0$, where x_0 is such that $a \leq x_0 \leq b$, then $\theta(x_0) = 0$.

Example

The equation $\frac{d^2y}{dx^2} + y = 0$ is of the type (3.3), where $P(x) = Q(x) = 1$ on I . The linearly dependent solutions $A_1 \sin x$ and $A_2 \sin x$ have the common zeros $x = n\pi$ ($n = 0, 1, 2, \dots$) and no other zeros.

3.3 The Separation and Comparison Theorems

Theorem 3.5: Let ϕ and θ be real linearly independent solutions of

$$\frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + Q(x)y = 0$$

on I . Between any two consecutive zeros of ϕ there is precisely one zero of θ .

Example

Consider the self-adjoint equation

$$\frac{d^2y}{dx^2} + y = 0.$$

The functions $\phi(x) = \sin x$ and $\theta(x) = \cos x$ are linearly independent solutions of the self-adjoint equation. Between any two consecutive zeros of one of these two linearly independent solutions there is indeed precisely one zero of the other solution.

Theorem 3.6 (Sturm's Fundamental Comparison Theorem)

Let ϕ_1, ϕ_2 be real solutions of

$$\frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + Q_1(x)y = 0$$

and

$$\frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + Q_2(x)y = 0$$

respectively on I . Let P have a continuous derivative and be such that $P(x) > 0$, and let Q_1 and Q_2 be continuous and such that $Q_2(x) > Q_1(x)$. Then if x_1 and x_2 are successive zeros of ϕ_1 on I , then ϕ_2 has at least one zero at some point of the open interval $x_1 < x < x_2$.

Example

Consider the equations

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0$$

and

$$\frac{d^2y}{dx^2} + \alpha^2 y = 0$$

where λ and α are constants such that $0 < \lambda < \alpha$. The functions ϕ_1 and ϕ_2 defined respectively by $\phi_1 = \sin \lambda x$ and $\phi_2(x) = \sin \alpha x$ are real solutions of these respective equations. Consecutive zeros of $\sin \lambda x$ are

$$\frac{n\pi}{\lambda} \text{ and } \frac{(n+1)\pi}{\lambda} \quad (n = 0, \pm 1, \pm 2, \dots).$$

By theorem 3.6, we are assured that $\sin \alpha x$ has at least one zero μ_n such that

$$\frac{n\pi}{\lambda} < \mu_n < \frac{(n+1)\pi}{\lambda} \quad (n = 0, \pm 1, \pm 2, \dots).$$

Specifically, $t = 0$ is a zero of both $\sin \lambda x$ and $\sin \alpha x$. The 'next' zero of $\sin \lambda x$ is $\frac{\pi}{\lambda}$, while the 'next' zero of $\sin \alpha x$ is $\frac{\pi}{\alpha}$; clearly $\frac{\pi}{\alpha} < \frac{\pi}{\lambda}$.

4.0 Stability

4.1 Preliminary Definitions

Consider the first-order system

$$x' = f(t, x) \tag{4.1}$$

where $f(t, x)$ is defined and continuous for all $t \geq t_0$ and x , and satisfies a Lipschitz condition in x in any bounded domain. We assume that solution of (4.1) given by $x = x(t; t_0, x_0)$, exists and it is unique.

In this section we seek the stability of solutions of (4.1).

Stability is concerned with the question as to whether solutions which are in some sense close to $x(t)$ at some instant will remain close for all subsequent times. Unstable solutions are thus extremely difficult to realize either experimentally or numerically, as an arbitrary small disturbance will eventually large deviation from the unstable solution. As an example, in applications such as automatic control theory, an important question is whether small changes in the initial conditions (input) lead to small changes (stability) or to large changes (instability) in the solution (output).

To study the stability of $x(t)$, consider the neighboring solution $y = y(t; t_0, y_0)$, where

$$y' = f(t, y), y(t_0) = y_0.$$

We are concerned here with the difference

$$y(t) - x(t) = u(t) \tag{4.2}$$

$$\begin{aligned} u' &= y' - x' \\ &= f(t, y) - f(t, x) \\ &= f(t, x(t) + u) - f(t, x(t)) \end{aligned}$$

If we let $F(t, u) = f(t, x(t) + u) - f(t, x(t))$, we have that

$$U' = F(t, u) \tag{4.3}$$

Here $F(t, 0) = 0$ for all $t \geq t_0$, and so the function $U = 0$ for all $t \geq t_0$ is a solution of (4.3). Consequently, the stability of $x(t)$ as a solution of (4.1) is reduced to the stability of the zero solution of (4.3).

Definition 1: The solution $u = 0$ of (4.3) is said to be stable, if for all $\varepsilon > 0$ and $t_1 \geq t_0$, there exists a $\delta(\varepsilon, t)$, such that $|u(t_1)| < \delta$ implies that $|z(t)| < \varepsilon$ for all $t \geq t_1$.

Definition 2: The solution $u = 0$ of (4.3) is said to be uniformly stable, if stable and $\delta = \delta(\varepsilon)$ is independent of t_1 .

Definition 3: The solution $u = 0$ of (4.3) is said to be asymptotically stable, if stable and $|z(t_1)| < \delta$ implies that $|z(t)| \rightarrow 0$ as $t \rightarrow \infty$.

Remark:

- (i) Definition 1 is also sometimes called Liapunor stability.
- (ii) $z = 0$ is said to be unstable if Definition 1 does not hold.

Example 1

Consider $z' + z = 0$,

Where $z = z_0 e^{-(t-t_0)}$ is the solution. Obviously the zero solution is uniformly and asymptotically stable.

Example 2

Consider $z' - z = 0$ with solution $z = z_0 e^{(t-t_0)}$. Here the zero solution is unstable since $|z| \rightarrow \infty$ as $t \rightarrow \infty$ for all $z \neq 0$.

Various methods exist for discussing the stability of linear or nonlinear system. In this section we shall treat one of the methods known as the Lyapunov second (direct) method.

Lyapunov's Direct Method

This method seeks a scalar function of u , which can be regarded as a measure of the 'energy' of the system (4.3), and then seeks to demonstrate that either this 'energy' decreases as $t \rightarrow \infty$, indicating stability, or it increases, indicating instability.

For simplicity, consider an autonomous system

$$U' = F(u) \quad (4.4)$$

where $F(u)$ is defined and continuous for all u , satisfies a Lipschitz condition in u in any bounded domain and is such that $F(0) = 0$. Let $V(u)$ be a scalar function of u , defined and continuous, with continuous partial derivatives for $|u| \leq c$ ($c > 0$), and such that $v(0) = 0$.

Definition 4: $V(u)$ is positive definite for $|u| \leq c$ if $V > 0$ for all $u \neq 0, |u| \leq c$.

Definition 5: $V(u)$ is negative definite for $|u| \leq c$ if $V < 0$ for all $u \neq 0, |u| \leq c$.

Definition 6: $V(u)$ is positive semi-definite for $|u| \leq k$ if $V \geq 0$ for all $u, |u| \leq k$.

Definition 7: $V(u)$ is negative semi-definite for $|u| \leq k$ if $V < 0$ for all $u, |u| \leq k$.

Example 3: The function

$$V = u_1^2 + u_2^2 + u_3^2$$

is positive definite.

Example 4: The function

$$V = (u_1 + u_2 + u_3)^2$$

is positive semi-definite.

Example 5: The function

$$V = -u_1^2 - u_2^2 - u_3^2$$

is negative definite.

Let $u(t)$ be a solution of (4.4) and consider the function $V(t) = V(u(t))$, thus the derivative of V along the trajectory $u(t)$ is

$$V' = \frac{d}{dt}V(u(t)) = (\nabla V)^T u' = (\nabla V)^T F(u)$$

where $\nabla V = \frac{\partial V}{\partial u_i}$ ($i = 1, 2, \dots, n$)

Theorem 1

- (i) Let $V(u)$ be positive definite and V' be negative semi-definite for $|z| \leq k$. Then the zero solution of (4.4) is uniformly stable.
- (ii) Let $V(z)$ be positive definite and V' be negative definite for $|z| \leq k$. Then the zero solution of (4.4) is uniformly and asymptotically stable.