MTS 101 ALGEBRA

Chapter 1

Elementary Theory of Sets

1.1 Introduction

The concept of set is fundamental in mathematics. A set is a well defined class of objects, such as the set of prime numbers, the set of points on a line, the set of mathematics teachers in a school and so on. The objects making up the set are called the elements, or members of the set. The elements of a set must have some characteristics in common, that is, we must be able to say precisely whether or not an object is a member of a given set. Sets are generally represented by upper case letters A,,B,C,... and their arbitrary members by lower case letters a,b,c,...

If X is a set and x is an element of X, we say that x belongs to X, and we write $x \in X$. If x does not belong to X, we write $x \notin X$. Given a set X and a statement p(x), there is a unique set Y whose elements are precisely those elements $x \in X$ for which p(x) is true. In symbols, we write

$$Y = \{x \in X : \text{p(x) is true}\}.$$

A set X is said to be finite if it has no elements, or if it contains countable number of distinct elements and the process of counting stops at a certain number, say k. The number k is called the cardinality of X, and we write $n(X) = k$. when $k = 0$, set X is said to be empty and X is called a null or a void or an empty set which is denoted by $\emptyset$. A set whose elements are not countable is otherwise called an infinite set.

Definition 1.1.1 *Let X and Y be sets. X is said to be a subset of Y if every element of x is an element of Y; that is, if*

$$x \in X \Rightarrow x \in Y.$$

If X is a subset of Y, we write $X \subseteq Y$. If Y contains some elements which are not present in X, we say that X is a proper subset of Y, and we write $X \subset Y$. The following statements are evident:

(i) $\emptyset \subset X$.

(ii) $X \subseteq X$.

(iii) If $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$.

(iv) $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$.

The statement given by (iv) is very important: the equality of two sets is usually established by showing that each of the two inclusions is valid.

The term universal or reference set is sometimes used for a set that contains all other sets in a given context and it is represented by $\mathcal{E}$.

The following sets frequently appear in mathematics:

$$\begin{aligned}
\mathcal{N} &= \{x : \text{x is a natural or counting number}, 1, 2, 3, 4, \ldots\} \\
\mathcal{Z} &= \{x : \text{x is an integer}, \ldots -3, -2, -1, 0, 1, 2, 3, \ldots\} \\
\mathcal{Z}^+ &= \{x \in \mathcal{Z} : x \geq 0\} \\
\mathcal{Z}^- &= \{x \in \mathcal{Z} : x \leq 0\} \\
\mathcal{Q} &= \{x : \text{x is a rational number}, x = \frac{a}{b}, a, b \in \mathcal{Z}, b \neq 0\} \\
\mathcal{Q}^+ &= \{x \in \mathcal{Q} : x \geq 0\}
\end{aligned}$$

$$\mathcal{Q}^- = \{x \in \mathcal{Q} : x \leq 0\}$$

$$\mathcal{R} = \{x : \text{x is a real number}\}$$

The symbols representing each of the sets should be noted carefully and also it should be observed that

$$\mathcal{N} \subseteq \mathcal{Z} \subseteq \mathcal{Q} \subseteq \mathcal{R}.$$

Definition 1.1.2 *Let X and Y be subsets of a reference set $\mathcal{E}$.*

(i) The union of X and Y, written $X \cup Y$, is the set

$$X \cup Y = \{x \in \mathcal{E} : x \in X \quad or \quad x \in Y\}.$$

(ii) The intersection of X and Y, written $X \cap Y$, is the set

$$X \cap Y = \{x \in \mathcal{E} : x \in X \quad and \quad x \in Y\}.$$

(iii) The difference of X and Y, written $Y - X$, is the set

$$Y - X = \{x \in \mathcal{E} : x \in Y \quad and \quad x \notin X\}.$$

If $X \subset Y$, then Y-X is called the complement of X with respect to Y. This is denoted by X' or X^c. X and Y are said to be disjoint if $X \cap Y = \emptyset$. It should be noted carefully that when $x \in X$ or $x \in Y$, it means that x belongs to at least one of X, Y and when $x \in X$ and $y \in Y$, it means that x belongs to both X and Y. The following two statements are immediate:

(i) For any two sets X, Y, $X \cap Y \subset X \subset X \cup Y$.

(ii) If $X \subset W$ and $Y \subset Z$, then $X \cup Y \subset W \cup Z$ and $X \cap Y \subset W \cap Z$.

The formal properties of the operations $\cup$ and $\cap$ are given in the following theorem.

Theorem 1.1.1 *Let X, Y, and Z be sets. Then*

(i) $X \cup X = X \cap X, \quad \forall X \qquad$ [idempotent]

(ii) $X \cup Y = Y \cup X$ *and* $X \cap Y = Y \cap X$ *[commutative]*

(iii) $X \cup (Y \cup Z) = (X \cup Y) \cup Z$ *and* $X \cap (Y \cap Z) = (X \cap Y) \cap Z$ *[associative]*

(iv) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ *and* $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ *[distributive]*

Proof: The verifications of (i)-(iii) are easy. To establish (iv), let

$$\begin{aligned} x \in X \cap (Y \cup Z) &\Leftrightarrow x \in X \quad \text{and} \quad x \in (Y \cup Z) \\ &\Leftrightarrow x \in X \quad \text{and} \quad [x \in Y \quad \text{or} \quad x \in Z] \\ &\Leftrightarrow [x \in X \quad \text{and} \quad x \in Y] \quad \text{or} \quad [x \in X \quad \text{and} \quad x \in Z] \\ &\Leftrightarrow x \in (X \cap Y) \quad \text{or} \quad x \in (X \cap Z) \\ &\Leftrightarrow x \in (X \cap Y) \cup (X \cap Z). \end{aligned}$$

Also, let

$$\begin{aligned} x \in X \cup (Y \cap Z) &\Leftrightarrow x \in X \quad \text{or} \quad x \in (Y \cap Z) \\ &\Leftrightarrow x \in X \quad \text{or} \quad [x \in Y \quad \text{and} \quad x \in Z] \\ &\Leftrightarrow [x \in X \quad \text{or} \quad x \in Y] \quad \text{and} \quad [x \in X \quad \text{or} \quad x \in Z] \\ &\Leftrightarrow x \in (X \cup Y) \quad \text{and} \quad x \in (X \cup Z) \\ &\Leftrightarrow x \in (X \cup Y) \cap (X \cup Z). \end{aligned}$$

Theorem 1.1.2 (De Morgan) *Let X and Y be subsets of a universal set $\mathcal{E}$. Then*

(i) $(X \cup Y)^c = X^c \cap Y^c$.

(ii) $(X \cap Y)^c = X^c \cup Y^c$.

Proof: (i) Let

$$\begin{aligned} x \in (X \cup Y)^c &\Leftrightarrow x \in \mathcal{E} \quad \text{and} \quad x \notin (X \cup Y) \\ &\Leftrightarrow x \in \mathcal{E} \quad \text{and} \quad [x \notin X \quad \text{or} \quad x \notin Y] \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \quad [x \in \mathcal{E} \quad \text{and} \quad x \notin X] \quad \text{and} \quad [x \in \mathcal{E} \quad \text{and} \quad x \notin Y] \\
&\Leftrightarrow \quad x \in X^c \quad \text{and} \quad x \in Y^c \\
&\Leftrightarrow \quad x \in (X^c \cap Y^c).
\end{aligned}$$

(ii) Also let

$$\begin{aligned}
x \in (X \cap Y)^c \quad &\Leftrightarrow \quad x \in \mathcal{E} \quad \text{and} \quad x \notin (X \cap Y) \\
&\Leftrightarrow \quad x \in \mathcal{E} \quad \text{and} \quad [x \notin X \quad \text{and} \quad x \notin Y] \\
&\Leftrightarrow \quad [x \in \mathcal{E} \quad \text{and} \quad x \notin X] \quad \text{or} \quad [x \in \mathcal{E} \quad \text{and} \quad x \notin Y] \\
&\Leftrightarrow \quad x \in X^c \quad \text{or} \quad x \in Y^c \\
&\Leftrightarrow \quad x \in (X^c \cup Y^c).
\end{aligned}$$

It is sometimes helpful to illustrate union, intersection, difference and complement of sets by means of Venn diagrams. Circles or ovals are drawn to represent the sets which are enclosed within a rectangle representing the universal set. Venn diagram is a useful tool in establishing basic and simple idenntities involving sets and also in solving two-set and three-set problems.

Theorem 1.1.3 *Let X, Y and Z be finite sets contained in the universal set $\mathcal{E}$. Then $X \cup Y \cup Z$ is finite and*

$$n(X \cup Y \cup Z) = n(X) + n(Y) + n(Z) - n(X \cap Y) - n(X \cap Z) - n(Y \cap Z) + n(X \cap Y \cap Z).$$

Proof: Suppose that X, Y and Z are finite sets contained in the universal set $\mathcal{E}$. Obviously, $X \cup Y \cup Z$ is finite and $\mathcal{E} = X \cup Y \cup Z$. From the venn diagram, we have

$$
\begin{aligned}
n(\mathcal{E}) &= n(X \cup Y \cup Z) \\
&= a + b + c + d + e + f + g \\
&= (a + b + c + d) + (c + d + e + f) + (b + d + f + g) \\
&\quad -(c + d + b + d + f + d) + d \\
&= (a + b + c + d) + (c + d + e + f) + (b + d + f + g) \\
&\quad -[(c + d) + (b + d) + (f + d)] + d \\
&= (a + b + c + d) + (c + d + e + f) + (b + d + f + g) \\
&\quad -(c + d) - (b + d) - (f + d) + d \\
&= n(X) + n(Y) + n(Z) - n(X \cap Y) - n(X \cap Z) \\
&\quad -n(Y \cap Z) + n(X \cap Y \cap Z) \\
\text{therefore} \quad n(X \cup Y \cup Z) &= n(X) + n(Y) + n(Z) - n(X \cap Y) \\
&\quad -n(X \cap Z) - n(Y \cap Z) + n(X \cap Y \cap Z) \qquad (1)
\end{aligned}
$$

If $Z = \emptyset$, (1) reduces to

$$n(X \cup Y) = n(X) + n(Y) - n(X \cap Y) \qquad (2)$$

Also if X and Y are disjoint, that is, $X \cap Y = \emptyset$, then (2) reduces to

$$n(X \cup Y) = n(X) + n(Y) \qquad (3)$$

Equations (1) and (2) are very useful in dealing with three-set and two-set problems respectively.

1.2 Worked Examples

Example 1.2.1 *If A, B and C are subsets of the universal set $\mathcal{E}$, represent the following sets on venn diagrams:*

(a) $(A \cup B) \cap (A \cup C)$;

(b) $(A \cap B) \cup (A \cap C)$;

(c) $A \cup (B \cap C)$;

(d) $A \cap (B \cup C)$.

What do you notice about:

(i) (a) and (c) ? (ii) (b) and (d) ?

Solution:

Example 1.2.2 *The universal set $\mathcal{E}$ is the set of all integers. A,B and C are subsets of $\mathcal{E}$ defined as follows:*

$$\begin{aligned} A &= \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\} \\ B &= \{x : 0 \leq x \leq 9\} \\ C &= \{x : -4 < x \leq 0\} \end{aligned}$$

(a) Write down the sets A', where A' is the complement of A with respect to $\mathcal{E}$.

(b) Find $B \cap C$.

(c) Find the members of the sets $B \cup C, A \cap B$ and $A \cap C$ and hence show that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Solution: (a)

$$\begin{aligned} A' &= \{x : x \in \mathcal{E} \quad \text{and} \quad x \notin A\} \\ &= \{\ldots, -7, -5, -3, -1, 1, 3, 5, 7, \ldots\} \end{aligned}$$

(b)

$$\begin{aligned} B &= \{x : 0 \leq x \leq 9\} \\ &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \\ C &= \{x : -4 < x \leq 0\} \\ &= \{-3, -2, -1, 0\} \\ B \cap C &= \{0\}. \end{aligned}$$

(c)

$$B \cup C = \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$\begin{aligned} A \cap B &= \{0,2,4,6,8\} \\ A \cap C &= \{-2,0\} \\ (A \cap B) \cup (A \cap C) &= \{-2,0,2,4,6,8\} \\ A \cap (B \cup C) &= \{-2,0,2,4,6,8\} = (A \cap B) \cup (A \cap C). \end{aligned}$$

Example 1.2.3 *In a certain class, 22 pupils take one or more of Chemistry, Economics and Government. 12 take Economics (E), 8 take Government (G) and 7 take Chemistry (C). Nobody takes Economics and Chemistry and 4 pupils take Economics and Government.*

(a) (i) Using set notation and the letters indicated above, write down the two statements in the last sentence

(ii) Draw a venn diagram to illustrate the information.

(b) How many pupils take

(i) both Chemistry and Government ?

(ii) Government only ?

Solution: (a) (i) Nobody takes Economics and Chemistry :

$$E \cap C = \emptyset \Rightarrow \quad n(E \cap C) = 0.$$

4 pupils take Economics and Government:

$$n(E \cap G) = 4.$$

(ii) Since $E \cap C = \emptyset$, it follows that $E \cap C \cap G = \emptyset$.

Given that $n(E \cup G \cup C) = 22$, we have

$$\begin{aligned} 8 + 4 + 4 - x + x + 7 - x &= 22 \\ \therefore \quad x &= 1. \end{aligned}$$

(b) From the venn diagram, we have that

(i) One pupil takes both Chemistry and Government.

(ii) Three pupils take Government only.

Example 1.2.4 *A school has 37 vacancies for teachers, out of which 22 are for English Language, 20 for History and 17 for Fine Art. Of these vacancies 11 are for both English Language and History, 8 for both History and Fine Art and 7 for English Language and Fine Art.*

Using a venn diagram, find the number of teachers who must be able to teach:

(a) all the three subjects;

(b) Fine Art only;

(c) English Language and History, but not Fine Art.

Solution: Let English Language, History and Fine Art be represented by E, H and F respectively. From the given data we have

$$\begin{aligned} n(E \cup H \cup F) &= 37 \\ n(E \cap H) &= 11 \\ n(H \cap F) &= 8 \\ n(E \cap F) &= 7. \end{aligned}$$

Let $n(E \cap H \cap F) = x$, then

$$\begin{aligned} n(E \cup H \cup F) &= 4 + x + 7 - x + x + 11 - x + 1 + x + 8 - x + 2 + x = 37 \\ \Rightarrow \quad 33 + x &= 37 \\ \therefore \quad x &= 4. \end{aligned}$$

It is clear from the venn diagram that

(a) 4 teachers must be able to teach all the three subjects.

(b) 6 teachers must be able to teach Fine Art only.

(c) 20 teachers must be able to teach English Language and History, but not Fine Art.

Example 1.2.5 *(a) In a certain school, 3 teachers teach Further Mathematics and 6 teach General Mathematics. If there are 7 Mathematics teachers in the school, how many of them teach both Futher Mathematics and General Mathematics ?*

(b) A newsagent sell three papers, the Guardian, the Punch and the Tribune. 70 customers buy the Guardian, 80 the Punch, and 90 the Tribune. 20 buy both the Guardian and the Punch, 25 the Punch and the Tribune, and 30 the Guardian and the Tribune. If 15 customers buy all the three papers, how many customers has the newsagent?

Solution: (a) Let F and G represent the sets of Further Mathematics and General Mathematics teachers respectively. From the given data we have $n(F) = 3, n(G) = 6, n(F \cup G) = 7$. We are to find $n(F \cap G)$. Using the formula for a two-set problem, we have

$$\begin{aligned} n(F \cup G) &= n(F) + n(G) - n(F \cap G) \\ \Rightarrow \quad 7 &= 3 + 6 - n(F \cap G) \\ \therefore \quad n(F \cap G) &= 9 - 7 \\ &= 2. \end{aligned}$$

Hence, 2 teachers teach both Further Mathematics and General Mathematics.

(b) Let G, P and T represent the set of customers who buy the Guardian, the Punch and the Tribune newspapers respectively. From the given data we have $n(G) = 70, n(P) = 80, n(T) = 90, n(G \cap P) = 20, n(P \cap T) = 25, n(G \cap T) = 30$ and $n(G \cap P \cap T) = 15$ It is required to find $n(G \cup P \cup T)$.

Using the formula for the three-set problem, we have

$$\begin{aligned}
n(G \cup P \cup T) &= n(G) + n(P) + n(T) - n(G \cap P) - n(P \cap T) - n(G \cap T) + n(G \cap P \cap T) \\
&= 70 + 80 + 90 - 20 - 25 - 30 + 15
\end{aligned}$$

$$7 = \quad 255 - 75$$

$$= \quad 180.$$

Thus the newsagent has 180 customers.

The information is represented in the venn diagram below.

Example 1.2.6 *If A and B are subsets of the reference set X, use set theoretic argument to show that:*

(a) $(A^c)^c = A$,

(b) $X - A^c = X \cap A$,

(c) $A \cup (A^c \cap B) = A \cup B$,

(d) $A \cap (A^c \cup B) = A \cap B$.

Solution: (a) Let

$$\begin{aligned}
x \in (A^c)^c &\Leftrightarrow x \notin A^c \\
&\Leftrightarrow x \in A \\
\therefore \quad (A^c)^c &= A.
\end{aligned}$$

(b) Let

$$\begin{aligned}
x \in (X - A^c) &\Leftrightarrow x \in X \quad \text{and} \quad x \notin A^c \\
&\Leftrightarrow x \in X \quad \text{and} \quad x \in A \\
&\Leftrightarrow x \in X \cap A \\
\therefore \quad X - A^c &= X \cap A.
\end{aligned}$$

(c) Let

$$\begin{aligned}
x \in A \cup (A^c \cap B) &\Leftrightarrow x \in A \quad \text{or} \quad x \in (A^c \cap B) \\
&\Leftrightarrow x \in A \quad \text{or} \quad [x \in A^c \quad \text{and} \quad x \in B] \\
&\Leftrightarrow x \in A \quad \text{or} \quad x \in B \\
&\Leftrightarrow x \in (A \cup B) \\
\therefore \quad A \cup (A^c \cap B) &= A \cup B.
\end{aligned}$$

(c) Let

$$\begin{aligned}
x \in A \cap (A^c \cup B) &\Leftrightarrow x \in A \quad \text{and} \quad x \in (A^c \cup B) \\
&\Leftrightarrow x \in A \quad \text{and} \quad [x \in A^c \quad \text{or} \quad x \in B] \\
&\Leftrightarrow x \in A \quad \text{and} \quad x \in B \\
&\Leftrightarrow x \in (A \cap B) \\
\therefore \quad A \cap (A^c \cup B) &= A \cap B.
\end{aligned}$$

1.3 Self Assessment Problems

SAP 1.3.1 *If A, B, C are subsets of X such that*

$$X = \{\alpha, \beta, \gamma, \lambda, \psi, \theta, \omega, \tau, \mu, \nu\}$$

$$
\begin{aligned}
A &= \{\alpha, \beta, \lambda, \theta, \omega, \tau, \mu\} \\
B &= \{\alpha, \beta, \gamma, \lambda, \omega, \mu\} \\
C &= \{\beta, \gamma, \lambda, \omega, \mu\}
\end{aligned}
$$

Find:

(a) A'

(b) B'

(c) C'

(d) $A' \cap B' \cap C$

(e) $A \cap B' \cap C'$

(f) $A' \cap B \cap C'$

(g) $A' \cap B' \cap C'$

and show that

(i) $(C \cup B) \cap A = (C \cap A) \cup (B \cap A)$

(ii) $(C \cap B) \cup A = (C \cup A) \cap (B \cup A)$.

Answer:

$$
\begin{aligned}
A' &= \{\gamma, \psi, \nu\} \\
B' &= \{\psi, \theta, \tau, \nu\} \\
C' &= \{\alpha, \psi, \theta, \tau, \nu\} \\
A' \cap B' \cap C &= \{\} = \emptyset \\
A \cap B' \cap C' &= \{\theta, \tau\} \\
A' \cap B \cap C' &= \{\} = \emptyset \\
A' \cap B' \cap C' &= \{\psi, \nu\}
\end{aligned}
$$

SAP 1.3.2 *The universal set $\mathcal{E}$ is the set of all integers and the subsets P, Q, R of $\mathcal{E}$ are given* *by*

$$\begin{aligned} P &= \{x : x \leq 0\} \\ Q &= \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\} \\ R &= \{x : -2 \leq x < 7\}. \end{aligned}$$

(a) Find $Q \cap R$

(b) Find R' where R' is the complement of R with respect to $\mathcal{E}$.

(c) Find $P' \cap R'$.

(d) List the members of $(P \cap Q)'$.

Answer:

$$\begin{aligned} Q \cap R &= \{-1, 1, 3, 5\} \\ R' &= \{\ldots, -7, -6, -5, -4, -3, 7, 8, \ldots\} \\ P' \cap R' &= \{7, 8, 9, 10, \ldots\} \\ (P \cap Q)' &= \{\ldots, -6, -4, -2, 0, 1, 2, 3, 4, 5, 6, \ldots\} \end{aligned}$$

SAP 1.3.3 *In a survey of 290 newspaper readers, 181 of them read the Daily Times, 142 read Guardian, 117 read the Punch and each reads at least one of the three papers. If 75 read the Daily Times and the Guardian, 60 read the Daily Times and the Punch and 54 read the Guardian and the Punch.*

(a) Draw a venn diagram to illustrate this information.

(b) How many readers read

(i) all three papers,

(ii) exactly two of the papers,

(iii) exactly one of the papers,

(iv) the Guardian alone ?

Answer: (a)

(b) (i) 39 (ii) 72 (iii) 179 (iv) 52

SAP 1.3.4 *After the registration of 100 freshmen, the following statistics were revealed: 60 were taking Mathematics, 44 were taking Physics, 30 were taking Chemistry, 15 were taking Physics and Chemistry, 6 were taking both Mathematics and Physics but not Chemistry, 24 were taking Mathematics and Chemistry, and 10 were taking all three subjects.*

(a) Show that 54 were enrolled in only one of the three subjects.

(b) Show that 35 were enrolled in at least two of them.

SAP 1.3.5 *Of a sample of 1000 students surveyed at the end of a term, 100 had applied for Lagos University, 80 for Ife, 75 for Benin, 30 had applied for Lagos and Benin, 20 for Lagos and Ife, 15 for Benin and Ife, and 5 had applied for all the three universities. Show that*

(a) 195 had applied for at least one of the universities.

(b) 805 had not applied.

(c) 55 had applied for Lagos only.

(d) 35 had applied for Benin only.

(e) 50 had applied for Ife only.

SAP 1.3.6 *A, B, C are subsets of a universal set X. Show that:*

(a) $A \cup (A \cap B) = A = A \cap (A \cup B)$

(b) $(A \cup B') \cap (A \cup B) = A$

(c) $A \cup (B \cap C) \neq (A \cap B) \cup C$

SAP 1.3.7 *A, B, C are subsets of a universal set X. Show that:*

(a) $(A \cup B)' = A' \cap B'$

(b) $(A \cap B)' = A' \cup B'$

(c) $(A')' = A$.

Hence or otherwise, show that

(d) $(A \cup B')' = A' \cap B$

(e) $(A' \cap B)' = A \cup B'$

(f) $(A' \cup B')' = A \cap B$

(g) $(A' \cap B')' = A \cup B$.

SAP 1.3.8 *A, B, C are subsets of a universal set X. Show that:*

(a) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

(b) $(A - B) - (A - C) = A \cap (C - B)$

(c) $(A - B) \cup (A - C) = A - (B \cap C)$

(d) $(A - B) \cap (A - C) = A - (B \cup C)$.

Chapter 2

Binary Operations

2.1 Introduction

Due to our familiarity with the four basic arithmetic operations of addition, subtraction, multiplication and division, if we are asked to add 2 and 3, our answer will be 5 and not 1 which could have been if we are to add in modulo 4. This shows that our answer depends on the rule of combination of the numbers involved.

A binary operation is said to have been performed if two elements of a set are combined according to a well defined rule to produce another element of the set. For example, $2, 3 \in \mathcal{N}$ and $2 + 3 = 5 \in \mathcal{N}$. Binary operations are often denoted by the symbols $\oplus, \otimes, \circ, \nabla, \diamond, *$ and so on.

Example 2.1.1 *A binary operation $*$ is defined over $\mathcal{R}$ the set of real numbers by*

$$x * y = x + y + xy.$$

(a) Evaluate:

$$(i) \quad 2 * 3 \qquad (ii) \quad 3 * 2 \qquad (iii) \quad 2 * (3 * 4) \qquad (iv) \quad (2 * 3) * 4.$$

(b) Solve the equations:

$$(i) \quad x * 3 = 19 \qquad (ii) \quad (x * 3) + (2 * x) = 40 \qquad (iii) \quad x * x = 48.$$

Solution: (a) (i) Given that $x * y = x + y + xy$, then

$$\begin{aligned} 2 * 3 &= 2 + 3 + 2 \times 3 \\ &= 5 + 6 \\ &= 11. \end{aligned}$$

(ii)

$$\begin{aligned} 3 * 2 &= 3 + 2 + 3 \times 2 \\ &= 5 + 6 \\ &= 11. \end{aligned}$$

(iii)

$$\begin{aligned} 2 * (3 * 4) &= 2 * (3 + 4 + 3 \times 4) \\ &= 2 * 19 \\ &= 2 + 19 + 2 \times 19 \\ &= 59. \end{aligned}$$

(iv)

$$\begin{aligned} (2 * 3) * 4 &= 11 * 4 \\ &= 11 + 4 + 11 \times 4 \\ &= 59. \end{aligned}$$

(b) (i)

$$\begin{aligned} x * 3 &= 19 \\ \Rightarrow \quad x + 3 + 3x &= 19 \\ \Rightarrow \quad 4x &= 16 \\ \therefore \quad x &= 4. \end{aligned}$$

(ii)

$$\begin{aligned}(x*3)+(2*3) &= 40\\ \Rightarrow\quad x+3+3x+2+x+2x &= 40\\ \Rightarrow\quad 7x &= 35\\ \therefore\quad x &= 5.\end{aligned}$$

(iii)

$$\begin{aligned}x*x &= 48\\ \Rightarrow\quad x+x+x^2 &= 48\\ \Rightarrow\quad x^2+2x-48 &= 0\\ \Rightarrow\quad (x-6)(x+8) &= 0\\ \Leftrightarrow\quad x &= 6 \quad \text{or} \quad x=-8.\end{aligned}$$

2.2 Properties of Binary Operations

Definition 2.2.1 (Closure Property) *A set A is said to be closed under a binary operation $*$ if for all $a, b \in A$, $a*b \in A$. For example, $\mathcal{N}$ is closed under the usual addition and multiplication.*

Example 2.2.1 *Let $X = \{x : 1 \leq x \leq 4\}$ and let a binary operation $*$ be defined on X such that for every $x, y \in X$,*

$$x * y = x \times_2 y$$

where $\times_2$ is the multiplication in modulo 2. Examine whether or not X is closed under $*$.

Solution: Given that

$$\begin{aligned}X &= \{x : 1 \leq x \leq 4\}\\ &= \{1,2,3,4\},\end{aligned}$$

Consider the table below.

$\times_2$	1	2	3	4
1	1	0	1	0
2	0	0	0	0
3	1	0	1	0
4	0	0	0	0

It is clear from the table that X is not closed under $*$ since $0 \notin X$.

Definition 2.2.2 (Commutative Property) *A binary operation $*$ over a set A is said to be commutative if for all $a, b \in A$, $a*b = b*a$. For example, the ordinary addition and multiplication are commutative over $\mathcal{Q}$ the set of rationals but the operations of ordinary subtraction and division are anti-commutative over $\mathcal{Q}$.*

Definition 2.2.3 (Associative Property) *A binary operation $*$ over a set A is said to be associative if for all $a, b, c \in A$,*

$$a * (b * c) = (a * b) * c.$$

For example, the operations of ordinary addition and multiplication are associative over $\mathcal{R}$ the set of real numbers but the operations of ordinary subtraction and division are not.

Definition 2.2.4 (Distributive Property) *A binary operation $*$ over a set A is said to be distributive over another binary operation $*'$ also defined over A if for all $a, b, c \in A$,*

$$a * (b *' c) = (a * b) *' (a * c).$$

For example, over $\mathcal{Z}$ the set of integers, the operation of usual multiplication is distributive over the operation of ordinary addition.

Also, the two operations $\cup$ and $\cap$ are distributive over each other since for every set A, B, C,

$$\begin{aligned} A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C). \end{aligned}$$

Example 2.2.2 *Let $\times_6$ denotes multiplication modulo 6 and let $+_6$ denotes addition modulo 6. Show that $\times_6$ is distributive over $+_6$.*

Solution: Let us pick 8, 11, and 15 from the set of natural numbers. By definition, $\times_6$ will be distributive over $+_6$ if and only if

$$a \times_6 (b +_6 c) = (a \times_6 b) +_6 (a \times_6 c) \qquad \forall a, b, c \in \mathcal{N}.$$

Now setting $a = 8, b = 11, c = 15$, we have

$$\begin{aligned} LHS &= a \times_6 (b +_6 c) \\ &= 8 \times_6 (11 +_6 15) \\ &= 8 \times_6 2 \\ &= 4 \text{mod} 6. \end{aligned}$$

Also put $a = 8, b = 11, c = 15$, we have

$$\begin{aligned} RHS &= (a \times_6 b) +_6 (a \times_6 c) \\ &= (8 \times_6 11) +_6 (8 \times_6 15) \\ &= 4 +_6 0 \\ &= 4 \text{mod} 6 \\ &= LHS. \end{aligned}$$

Thus, $\times_6$ is distributive over $+_6$. The reader should also try the verification using other natural numbers.

Definition 2.2.5 (Identity Element) *Let $*$ be a binary operation over a set A. An element $e \in A$ is said to be an identity or a neutral element if for all $a \in A$,*

$$a * e = e * a = a.$$

For example in $\mathcal{R}$, the set of real numbers, 0 and 1 are the additive and multiplicative identities respectively since

$$\begin{aligned} 0 + a &= a + 0 = a \\ 1 \times b &= b \times 1 = b \quad \forall\ a, b \in \mathcal{R}. \end{aligned}$$

Definition 2.2.6 (Inverse Element) *Let $*$ be a binary operation over a set A. An element $b \in A$ is said to be an inverse of an element $a \in A$ if for all $a \in A$,*

$$a * b = b * a = e$$

where e is the identity element of the set A. The inverse of a if it exists is generally denoted by a^{-1}. For example in $\mathcal{R}$, the set of real numbers,

$$a + (-a) = (-a) + a = 0 \quad \forall\ a \in \mathcal{R}$$

also for all $(a \neq 0) \in \mathcal{R}$,

$$a \times a^{-1} = a^{-1} \times a = 1.$$

Thus, (-a) is the additive inverse of an element $a \in \mathcal{R}$ and a^{-1} is the multiplicative of a nonzero element $a \in \mathcal{R}$.

It should be noted that the identity element and the inverse element in a set under a given binary operation are unique. Also, the existence of an inverse element depends on the existence of an identity element in a given set under a given binary operation.

2.3 Worked Examples

Example 2.3.1 *A binary operation $*$ is defined on $\mathcal{R}$ the set of real numbers by*

$$x * y = \frac{xy}{x+y}, \quad x + y \neq 0.$$

(a) Show that $*$ is commutative and associative.

(b) Obtain the values of:

$$(i) \quad 3 * -5 \qquad (ii) \quad 2 * 7 \qquad (iii) \quad -3 * (5 * -7) \qquad (iv) \quad \frac{2}{3} * \frac{-5}{7}.$$

(c) Obtain the values of x for which

$$(i) \quad 3 * x = 5/7 \qquad (ii) \quad (x * 3) + (4 * x) = 7.$$

Solution: (a)

$$\begin{aligned} x * y &= \frac{xy}{x+y} \\ &= \frac{yx}{y+x} \\ &= y * x. \end{aligned}$$

The commutativity of $*$ follows.

For associativity, we must show that for all $x, y, z \in \mathcal{R}$, $x * (y * z) = (x * y) * z$. Now put

$$\begin{aligned} LHS &= x * (y * z) \\ &= x * \frac{yz}{y+z} \\ &= \frac{xyz}{y+z} \div \left(x + \frac{yz}{y+z} \right) \\ &= \frac{xyz}{y+z} \times \frac{y+z}{xy+xz+yz} \\ &= \frac{xyz}{xy+xz+yz}. \\ RHS &= (x * y) * z \end{aligned}$$

$$
\begin{aligned}
&= \frac{xy}{x+y} * z \\
&= \frac{xyz}{x+y} \div \left(\frac{xy}{x+y} + z\right) \\
&= \frac{xyz}{x+y} \times \frac{x+y}{xy+xz+yz} \\
&= \frac{xyz}{xy+xz+yz} \\
&= LHS.
\end{aligned}
$$

The associativity of $*$ then follows.

(b) (i)

$$
\begin{aligned}
3 * -5 &= \frac{3 \times -5}{3-5} \\
&= \frac{15}{2}.
\end{aligned}
$$

(ii)

$$
\begin{aligned}
2 * 7 &= \frac{2 \times 7}{2+7} \\
&= \frac{14}{9}.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
-3 * (5 * -7) &= -3 * \left(\frac{5 \times -7}{5-7}\right) \\
&= -3 * \frac{35}{2} \\
&= \frac{-3 \times \frac{35}{2}}{-3 + \frac{35}{2}} \\
&= -\frac{105}{29}.
\end{aligned}
$$

(iv)

$$
\begin{aligned}
\frac{2}{3} * \frac{-5}{7} &= \frac{\frac{2}{3} \times \frac{-5}{7}}{\frac{2}{3} - \frac{5}{7}} \\
&= \frac{-10}{21} \div \frac{-1}{21} \\
&= \frac{-10}{21} \div \frac{-21}{1} \\
&= 10.
\end{aligned}
$$

(c) (i)

$$\begin{aligned} x * y &= \frac{5}{7} \\ \Rightarrow \quad \frac{3x}{3+x} &= \frac{5}{7} \\ \Rightarrow \quad 21x - 5x &= 15 \\ \therefore \quad x &= \frac{15}{16}. \end{aligned}$$

(ii)

$$\begin{aligned} (x * 3) + (4 * x) &= 7 \\ \Rightarrow \quad \frac{3x}{3+x} + \frac{4x}{4+x} &= 7 \\ \Rightarrow \quad 12x + 3x^2 + 4x^2 + 12x &= 7(x^2 + 7x + 12) \\ \Rightarrow \quad 49x - 24x &= -84 \\ \therefore \quad x &= -\frac{84}{25}. \end{aligned}$$

Example 2.3.2 *The operation $*$ is defined over $\mathcal{R}$ the set of real numbers by*

$$p * q = p + q - \frac{1}{2}pq.$$

(a) Show that $*$ is commutative and associative.

(b) Find the identity element for the operation $*$.

(c) Find the inverse (under $*$) of the real number p, stating any value of p for which no inverse exists.

(d) Determine whether or not

$$p * (q + r) = (p * q) + (p * r), \qquad \forall\ p, q, r \in \mathcal{R}.$$

Solution: (a)

$$p * q \;=\; p + q - \frac{1}{2}pq$$

$$\begin{aligned} &= q + p - \frac{1}{2}qp \\ &= q * p. \end{aligned}$$

The commutativity of $*$ is immediate.

For associativity, let $p, q, r \in \mathcal{R}$. Then

$$\begin{aligned} p * (q * r) &= p * (q + r - \frac{1}{2}qr) \\ &= p + (q + r - \frac{1}{2}qr) - \frac{1}{2}p(q + r - \frac{1}{2}qr) \\ &= p + q + r - \frac{1}{2}qr - \frac{1}{2}pq - \frac{1}{2}pr + \frac{1}{4}pqr. \\ (p * q) * r &= (p + q - \frac{1}{2}pq) * r \\ &= (p + q - \frac{1}{2}pq) + r - \frac{1}{2}(p + q - \frac{1}{2}pq)r \\ &= p + q + r - \frac{1}{2}qr - \frac{1}{2}pq - \frac{1}{2}pr + \frac{1}{4}pqr. \\ &= p * (q * r). \end{aligned}$$

The associativity of $*$ is immediate.

(b) Suppose that $e \in \mathcal{R}$ is the identity element for $*$, then

$$\begin{aligned} p * e &= e * p = p \\ \Rightarrow \quad p + e - \frac{1}{2}pe &= p \\ \Rightarrow \quad e(2 - p) &= 0 \\ \Leftrightarrow \quad e &= 0 \qquad \text{[provided that } p \neq 2]. \end{aligned}$$

(c) Suppose that k is the inverse of p under $*$, then

$$\begin{aligned} p * k &= k * p = e = 0 \\ \Rightarrow \quad p + k - \frac{1}{2}pk &= 0 \\ \Rightarrow \quad k(p - 2) &= 2p \\ \therefore \quad k &= \frac{2p}{p - 2}. \end{aligned}$$

No inverse will exist if $p - 2 = 0$ that is when $p = 2$.

(d)

$$\begin{aligned} p * (q + r) &= p + (q + r) - \frac{1}{2}p(q + r) \\ &= p + q + r - \frac{1}{2}pq - \frac{1}{2}pr. \\ (p * q) + (p * r) &= p + q - \frac{1}{2}pq + p + r - \frac{1}{2}pr \\ &= 2p + qr - \frac{1}{2}pq - \frac{1}{2}pr \\ &\neq p * (q + r). \end{aligned}$$

Example 2.3.3 *Let X be a nonempty set with associative binary operation $\triangle$. Let $x, y, z \in X$. Suppose x commutes with y and z, show that x commutes also with $y \triangle z$.*

Solution: Given that x commutes with y and z, then $x \triangle y = y \triangle x$ and $x \triangle z = z \triangle x$. Now,

$$\begin{aligned} x \triangle (y \triangle z) &= (x \triangle y) \triangle z \qquad \text{[since } \triangle \text{ is associative]} \\ &= (y \triangle x) \triangle z \\ &= y \triangle (x \triangle z) \\ &= y \triangle (z \triangle x) \\ &= (y \triangle z) \triangle x. \end{aligned}$$

Evidently, x commutes with $y \triangle z$.

Example 2.3.4 *Let J be a nonempty set with associative binary operation $\triangle$. Show that the binary operation ∇ given by*

$$x \nabla y = x \triangle j \triangle y$$

is also associative. If $\triangle$ is commutative, is ∇ commutative ?

Solution: Given that $\triangle$ is associative binary operation, let $x, y, z \in J$. Then

$$x \nabla (y \nabla z) = x \nabla (y \triangle j \triangle z)$$

$$
\begin{aligned}
&= x\triangle j\triangle(y\triangle j\triangle z) \\
&= x\triangle j\triangle y\triangle j\triangle z. \\
(x\nabla y)\nabla z &= (x\triangle j\triangle y)\nabla z \\
&= (x\triangle j\triangle y)\triangle j\triangle z \\
&= x\triangle j\triangle y\triangle j\triangle z \\
&= x\nabla(y\nabla z).
\end{aligned}
$$

Accordingly, ∇ is associative over J.

Lastly, suppose that $\triangle$ is commutative, then ∇ will be commutative if and only if for all $x, y \in J$, we can show that $x\nabla y = y\nabla x$. To this end,

$$
\begin{aligned}
x\nabla y &= x\triangle j\triangle y \\
&= x\triangle(j\triangle y) \\
&= x\triangle(y\triangle j) \\
&= (x\triangle y)\triangle j \\
&= (y\triangle x)\triangle j \\
&= y\triangle(x\triangle j) \\
&= y\triangle(j\triangle x) \\
&= y\triangle j\triangle x \\
&= y\nabla x.
\end{aligned}
$$

The required result then follows.

Example 2.3.5 *Consider the set I of ordered pairs*

$$I = \{(m, n) : m,\ n \text{ are natural numbers}\}.$$

An operation $\oplus$ is defined on I by

$$(a, b) \oplus (c, d) = (a + c, b + d).$$

Show that this operation is commutative and associative.

Any two elements (a,b), (c,d) in I are to be considered equal if and only if $a + d = b + c$. Show that any element of the form (n,n) may be regarded as a neutral element with respect to $\oplus$.

Given that (r,s) is an inverse of (p,q), find the relationship between p, q, r, s. Hence find an inverse for the element (7,5) and an inverse for the element (m,n).

Solution: Let (a,b), (c,d) and (e,f) be any elements of I. Then

$$\begin{aligned} (a, b) \oplus (c, d) &= (a + c, b + d) \\ &= (c + a, d + b) \\ &= (c, d) \oplus (a, b). \end{aligned}$$

This shows that $\oplus$ is commutative.

Also,

$$\begin{aligned} (a, b) \oplus ((c, d) \oplus (e, f)) &= (a, b) \oplus (c + e, d + f) \\ &= (a + (c + e), b + (d + f)) \\ &= ((a + c) + e, (b + d) + f) \\ &= (a + c, b + d) \oplus (e, f) \\ &= ((a, b) \oplus (c, d)) \oplus (e, f). \end{aligned}$$

This establishes the associativity of $\oplus$.

Next, given that $(a, b) = (c, d)$ if and only if $a+d = b+c$, let $(e, f) \in I$ be an identity element. Then for any $(a, b) \in I$,

$$(a, b) \oplus (e, f) = (e, f) \oplus (a, b) = (a, b)$$

$$
\begin{aligned}
\Rightarrow \quad (a+e, b+f) &= (a, b) \\
\Leftrightarrow \quad a+e+b &= b+f+a \\
\Leftrightarrow \quad e &= f = n.
\end{aligned}
$$

Hence, any element of the form (n,n) where n is a natural number is a neutral element with respect to $\oplus$.

Lastly, given that (r,s) is an inverse of (p,q), then

$$
\begin{aligned}
(r, s) \oplus (p, q) &= (p, q) \oplus (r, s) = (n, n) \\
\Rightarrow \quad (p+r, q+s) &= (n, n) \\
\Leftrightarrow \quad p+r+n &= q+s+n \\
\Leftrightarrow \quad p+r &= q+s \qquad (1)
\end{aligned}
$$

which is the required relationship between p, q, r and s.

To obtain the inverse of (7,5), put $(p, q) = (7, 5)$ in equation (1) to obtain

$$7 + r = 5 + s$$

This equation is obviously satisfied by $r = 5$ and $s = 7$. Hence, (5,7) is the inverse of (7,5). Using the same procedure, we obtain (n,m) as the inverse of (m,n).

Example 2.3.6 *Two binary operations $\oplus$ and $\otimes$ over the universal set $\mathcal{E}$ are defined by*

$$
\begin{aligned}
A \oplus B &= A \cup B, \\
A \otimes B &= A \cap B \quad \forall\ A, B \subset \mathcal{E}.
\end{aligned}
$$

Show that:

(a) $\oplus$ is both commutative and associative,

(b) $\otimes$ is both commutative and associative,

(c) $\oplus$ is distributive over $\otimes$,

(d) $\otimes$ is distributive over $\oplus$.

Solution: For every $A, B, C \subset \mathcal{E}$, we have

(a)

$$\begin{aligned} A \oplus B &= A \cup B \\ &= B \cup A \\ &= B \oplus A. \\ A \oplus (B \oplus C) &= A \cup (B \cup C) \\ &= (A \cup B) \cup C \\ &= (A \oplus B) \oplus C. \end{aligned}$$

(b)

$$\begin{aligned} A \otimes B &= A \cap B \\ &= B \cap A \\ &= B \otimes A. \\ A \otimes (B \otimes C) &= A \cap (B \cap C) \\ &= (A \cap B) \cap C \\ &= (A \otimes B) \otimes C. \end{aligned}$$

(c)

$$\begin{aligned} A \oplus (B \otimes C) &= A \cup (B \cap C) \\ &= (A \cup B) \cap (A \cup C) \\ &= (A \oplus B) \otimes (A \oplus C). \end{aligned}$$

(d)

$$\begin{aligned} A \otimes (B \oplus C) &= A \cap (B \cup C) \\ &= (A \cap B) \cup (A \cap C) \\ &= (A \otimes B) \oplus (A \otimes C). \end{aligned}$$

2.4 Self Assessment Problems

SAP 2.4.1 *A binary operation $*$ is defined over the set $\mathcal{R}$ of real numbers by*

$$x * y = x + y - x^2 y.$$

(a) Determine whether or not $*$ is commutative and associative.

(b) Evaluate:

$$(i) \quad 2 * 3 \qquad (ii) \quad -5 * 4 \qquad (iii) \quad 3 * (4 * 5).$$

(c) Find the value(s) of x for which:

$$(i) \quad 4 * x = 34 \qquad (ii) \quad (3 * x) + (x * 3) = 8.$$

Answer:

(b) -7, -101, 571

(c) -2, -1/3 or -2.

SAP 2.4.2 *The function f is defined by*

$$f : x \to 3x - 2, \qquad x \in \mathcal{R}.$$

(a) The binary operation $\circ$ on the set $\mathcal{R}$ is such that

$$f(p \circ q) = f(p) \times f(q) \qquad \forall\ p, q \in \mathcal{R}.$$

(i) Show that $p \circ q = 3pq - 2p - 2q + 2$.

(ii) Show that $\circ$ *is commutative and associative.*

(iii) Find the identity element for the operation.

(iv) Find the inverse (under $\circ$*) of the real number* p*, stating any value of* p *for which no inverse exists.*

(b) Another binary operation $\bullet$ *on the set* $\mathcal{R}$ *is such that*

$$f(p \bullet q) = \frac{f(p)}{f(q)}, \quad f(q) \neq 0 \quad \forall\ p, q \in \mathcal{R}.$$

(i) Show that

$$p \bullet q = \frac{p + 2q - 2}{3q - 2}, \quad q \neq \frac{2}{3}.$$

(ii) Show that $\bullet$ *is neither commutative nor associative.*

(iii) Determine whether or not

$$p \bullet (q \circ r) = (p \bullet q) \circ (p \bullet r) \quad \forall\ p, q, r \in \mathcal{R}.$$

Answer:

(a) (iii) 1 (iv) $\frac{2p-1}{3p-2}$, $p = 2/3$.

SAP 2.4.3 *Let* S *be the set of all ordered pairs* $x = (x_1, x_2)$ *with* x_1 *and* x_2 *real numbers. A binary operation* $*$ *is defined on* S *by*

$$a * b = (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1).$$

Show that this operation is commutative and associative.

Determine the identity element for this operation, and also the inverse of any element x*. Hence solve:*

$$a * x = b \qquad where a = (3, 4), b = (5, 6).$$

Answer:

identity $= (1,0)$, $x^{-1} = \left(\frac{x_1}{x_1^2+x_2^2}, \frac{-x_2}{x_1^2+x_2^2}\right)$, $x = \left(\frac{39}{25}, \frac{-2}{25}\right)$.

SAP 2.4.4 *Let X be a nonempty set with associative binary operation $\circ$. If e and f are elements of X such that $x \circ e = x$ and $f \circ x = x$ for all x in X, show that $e = f$.*

Furthermore, if $x \circ y = e = z \circ x$, show that $y = z$.

SAP 2.4.5 *For any two subsets X and Y of a universal set $\mathcal{E}$, the operation $\bullet$ is defined by*

$$X \bullet Y = (X \cap Y') \cup (Y \cap X'),$$

where X', Y' denote the complements of X and Y respectively. Show that:

(a) the operation is commutative;

(b) the empty set $\emptyset$ is the identity element for $\bullet$;

(c) every element is its own inverse.

SAP 2.4.6 *Find the identity element, if it exists, and the inverse of 5 when each of the following operations is defined on $\mathcal{R}$ the set or real numbers.*

(a) $p * q = p + q$

(b) $p * q = pq$

(c) $p * q = p + q + pq$

(d) $p * q = pq + 2p + 2q$

(e) $p * q = \sqrt{pq}$

(f) $p * q = \frac{p}{q} + \frac{q}{p}$

(g) $p * q = \frac{p}{q} - p$.

Answer:

(a) 0,-5 (b) 1,1/5 (c) 0,-5/6 (d) no identity (e) no identity (f) no identity (g) 1/2,5/3.