1 Quadratic Equations

1.1 Roots of a quadratic equation

The general form of a quadratic equation is

$$ax^2 + bx + c = 0 \tag{1.1}$$

where a, b, c are constants and $a \neq 0$.
The roots of the quadratic equation (1.1) is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}$$

where $D = b^2 - 4ac$.

The nature of the roots of a quadratic equation is determined by the value of D.

(i) If $D > 0$, the equation will have two different real roots.

(ii) If $D = 0$, the equation has two equal roots.

(iii) If $D < 0$, the equation has complex roots.

1.2 Sum and product of the roots

Let α and β be the roots of the quadratic equation (1.1), then it is equivalent to the equation

$$(x - \alpha)(x - \beta) = 0$$

or

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \tag{1.2}$$

Dividing (1.1) through by a, we have

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \tag{1.3}$$

Comparing (1.2) and (1.3), we obtain

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a} \tag{1.4}$$

Using (1.4), we can find the sum and the product of the roots directly from the coefficients in the quadratic equation (1.1).

Example 1. If the roots of $x^2 - x + 1 = 0$ are α and β, find $\alpha + \beta$ and $\alpha\beta$.

Comparing the given equation with (1.1), $a = 1$, $b = -1$, $c = 1$.
Hence

$$\alpha + \beta = 1$$

and

$$\alpha\beta = 1$$

.

Example 2. Construct an equation with roots $\sqrt{5} + 2$, $\sqrt{5} - 2$.

Let $\alpha = \sqrt{5} + 2$ and $\beta = \sqrt{5} - 2$.
Then

$$\alpha + \beta = \sqrt{5} + 2 + \sqrt{5} - 2 = 2\sqrt{5}$$
$$\alpha\beta = (\sqrt{5} + 2)(\sqrt{5} - 2) = 1$$

Using (1.2), the equation is

$$x^2 - 2\sqrt{5}x + 1 = 0.$$

Example 3. If α and β are the roots of the equation $ax^2 + bx + c = 0$, obtain in terms of a,b and c the values of (i) $\frac{1}{\alpha} + \frac{1}{\beta}$ (ii) $\alpha - \beta$.

We express (i) and (ii) in terms of $\alpha + \beta$ and $\alpha\beta$.

(i) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{-\frac{b}{a}}{\frac{c}{a}} = -\frac{b}{c}$.

(ii) $\alpha - \beta = \pm\sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = \pm\sqrt{(-\frac{b}{c})^2 - 4\frac{c}{a}} = \pm\sqrt{\frac{b^2}{a^2} - \frac{4c}{a}}$

Hence, $\alpha - \beta = \pm\frac{1}{a}\sqrt{b^2 - 4ac}$.

Example 4. If α,β are the roots of the equation $3x^2 - x - 5 = 0$, form the equation whose roots are $2\alpha - \frac{1}{\beta}$, $2\beta - \frac{1}{\alpha}$.

From the given equation, $a = 3$, $b = -1$ and $c = -5$. Thus,

$$\alpha + \beta = \frac{1}{3}, \quad \alpha\beta = -\frac{5}{3}.$$

Given the roots $2\alpha - \frac{1}{\beta}$ and $2\beta - \frac{1}{\alpha}$,

$$2\alpha - \frac{1}{\beta} + 2\beta - \frac{1}{\alpha} = 2(\alpha+\beta) - (\frac{\alpha+\beta}{\alpha\beta})$$
$$= 2(\frac{1}{3}) - (+\frac{1}{3} \times -\frac{3}{5})$$
$$= \frac{2}{3} + \frac{1}{5}$$
$$= \frac{10+3}{15}$$
$$= \frac{13}{15}$$

$$(2\alpha - \frac{1}{\beta})(2\beta - \frac{1}{\alpha}) = 4\alpha\beta + \frac{1}{\alpha\beta} - 4$$
$$= -\frac{169}{15}$$

Therefore the required equation is

$$x^2 - \frac{13}{15}x - \frac{169}{15} = 0$$

that is

$$15x^2 - 13x - 169 = 0.$$

Exercise: One root of the equation $2x^2 + bx + c = 0$ is three times the other root. Show that $3b^2 = 32c$.

2 Cubic Equations

2.1 Introduction

The general form of a cubic equation is

$$ax^3 + bx^2 + cx + d = 0 \tag{2.1}$$

where a,b,c and d are constants, $a \neq 0$.

Equation (2.1) is also expressible as

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0 \tag{2.2}$$

If α,β and γ are roots of the cubic equations (2.1), (2.2), then

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} \equiv (x-\alpha)(x-\beta)(x-\gamma)$$
$$= x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma$$

Thus comparing coefficients,

$$\alpha + \beta + \gamma = -\frac{b}{a}$$
$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$
$$\alpha\beta\gamma = -\frac{d}{a}.$$

Thus the equation whose roots are α,β,γ is

$$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma = 0 \tag{2.3}$$

2.2 Useful identities and examples

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$
$$\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma.$$

Example 2.1. If α,β and γ are the roots $x^3 - 7x + 1 = 0$, find the equation whose roots are α^2,β^2,γ^2.

Solution: From the given equation

$$\alpha + \beta + \gamma = 0$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = -7$$

$$\alpha\beta\gamma = 1$$

Thus,

$$\begin{aligned}\alpha^2 + \beta^2 + \gamma^2 &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)\\ &= 0 - 2(-7)\\ &= 14\end{aligned}$$

$$\begin{aligned}\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 &= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2(\alpha\beta^2\gamma + \alpha^2\beta\gamma + \alpha\beta\gamma^2)\\ &= (-7)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)\\ &= 49 - 2(1)(0)\\ &= 49\end{aligned}$$

$$\alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = 1^2 = 1.$$

Hence, the required equation is

$$x^3 - 14x^2 + 49 - 1 = 0.$$

Exercise

1. If α,β and γ are the roots of $x^3 - 3x + 1 = 0$, find the value of $\alpha^3 + \beta^3 + \gamma^3$.

2. If α,β and γ are the roots of the equation $ax^3 + bx^2 + cx + d = 0$, where $a \neq 0$, show that

$$(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 = \frac{(c^2 - 2bd)}{a^2}.$$

3 Simultaneous Equations

3.1 Simultaneous linear equations in two variables

The general form of simultaneous linear equations in two variables is
$ax + by = c$
$dx + ey = f$
where x,y are variables, a,b,c,d,e,f are constants.

Various methods exist for solving these equations for x and y. These are:

(i) elimination method

(ii) substitution method

(iii) matrix method and

(iv) graphical method.

The reader is encouraged to find out.

3.2 Simultaneous equations, atleast one non-linear

The general form of simultaneous equations in which one is linear one is quadratic is
$ax + bx = c$
$dx^2 + exy + fy^2 = g$
where x,y are variables, and a,b,c,d,e,f,g are arbitrary constants.

Example: Solve the simultaneous equations for x and y.

$$x^2 + y^2 = 25 \tag{1}$$

$$x + 3y = 5 \tag{2}$$

From (2), x=5-3y.
Substitute for x in (2),
$(5 - 3y)^2 + y^2 = 25$
$y^2 - 3y = 0$
$y(y - 3) = 0$
either $y = 0$ or $y = 3$.
Hence, $x = 5$ when $y = 0$ or
$x = -4$ when $y = 3$.