MCE 401: FLUID MECHANIC (IV)-3UNITS

Laminar boundary layers. The Navier–Stokes equations and their approximation. Very slow motion and lubrication. Flat plate flow. Integral analysis. Similarity solutions. Approximate methods of solutions. Pressure gradient effects and separation .Transition: Stability of laminar boundary layers to disturbances. Onset and development of turbulence. Turbulent boundary layers, mean flow and fluctuations. Reynolds stress, eddy viscosity and mixing length. The log – law smooth flat-plate flow and drag laws. Simple calculation methods, Free shear layers, Laminar and Turbulent mixing layers, jets and wakes. Compressible flow, Mach number and mach waves. Shock waves and heat transfer, Isentropic flow through ducts of varying area. Converging-diverging flow with normal shocks. Flow through constant area ducts with and without friction. Unsteady gas dynamic and the method of characteristics blast waves and detonation.

Textbooks

1. I. H. Shames, Mechanics of Fluids, Mc-Graw-Hill International Book Company.

   B. S. Massy, Mechanic of Fluids

2. W. J. Duncan, A. S. Thom A. D. Young, Mechanics of Fluids.

3. Gasiorek, Swaffield, Douglas, Mechanics of Fluids.

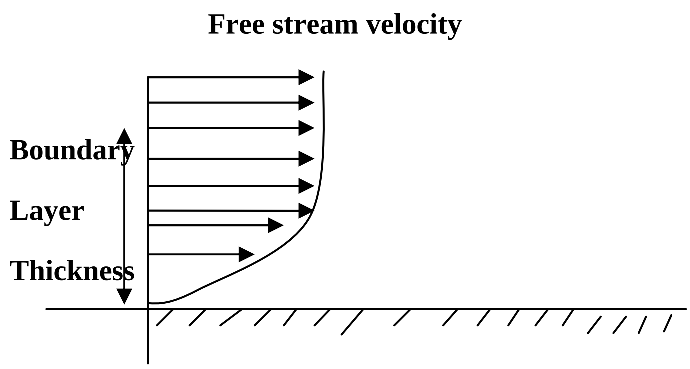
4. V. L Streeters, E. B. Wyles, Fluids mechanics

BOUNDARY LAYER FLOW

Introduction

In flow of real fluids

- No discontinuity of velocity
- No slip at solid surface



Boundary layer is the region in which the velocity increases rapidly from zero and approaches the velocity of the main stream. Boundary layer is usually thin and velocity gradient is high and shear stresses are important. In 1904 Ludwig Prandtl suggested that the flow over an object can be considered in 2 patterns.

(1) The boundary layer where shear stresses are important.

(2) Beyond the boundary layer where velocity gradients are small and the effect of viscosity is negligible. The flow there is essentially that of an ideal fluid.

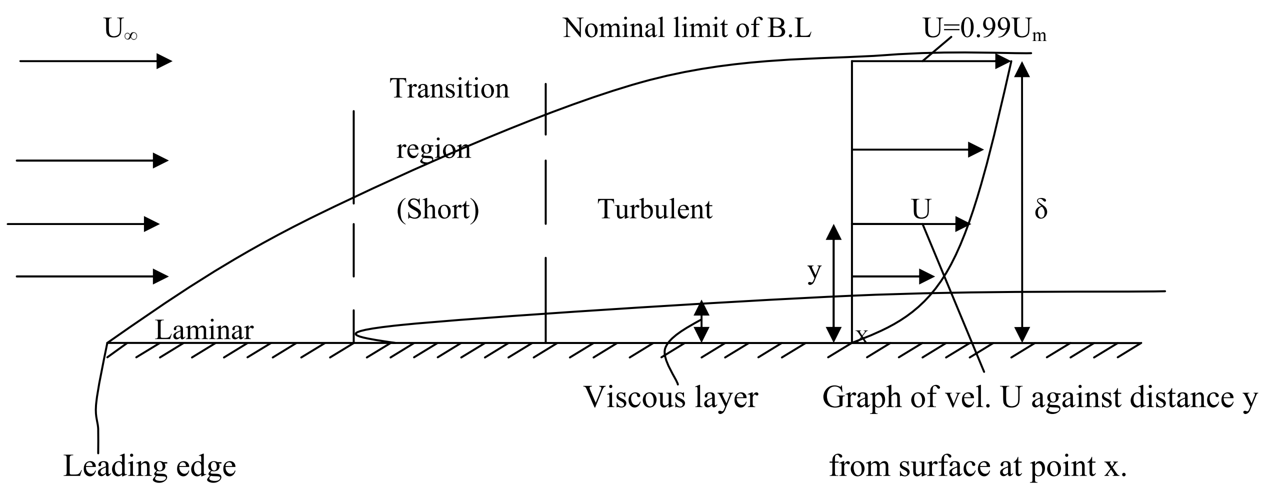
Description of the boundary layer

The simplest boundary layer to study is that formed in the flow along one side of a thin, smooth, flat plate parallel to the direction of the oncoming fluid.

- No other solid surface is near
- Pressure is assumed uniform
- For ideal fluid, no velocity gradient
- Velocity gradients in a real fluid are due to viscous action near the surface.

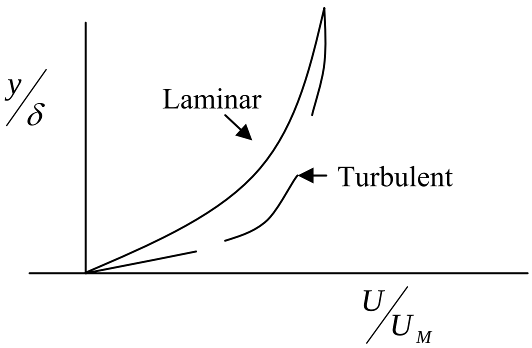
The oncoming stream with velocity, U_∞ is retarded in the neighborhood of the surface and the boundary layer begins at the leading edge of the plate and increases in thickness as more fluid is slowed down.

F.g1.1 Boundary layer on the flat plate



The thickness of the boundary layer may be taken as that distance from the surface at which the velocity reaches 99% of the velocity of the main stream. The flow in the first part of the boundary layer is entirely laminar but developed into transition and finally into turbulent boundary layer. At any distance x, from the leading edge of the plate the boundary layer thickness δ is very small compared with x.

F.g1.2: Typical velocity distribution in laminar and turbulent boundary layers on a flat plate.



$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \quad \dots\dots\dots\dots\dots\dots(1.1)$$

In laminar boundary layer, $\delta \propto x^{0.5}$

In turbulent boundary layer, $\delta \propto x^{0.8}$

When $\frac{\partial P}{\partial x} = 0$

Reynolds number, $R_e = \frac{Inertia\ force}{Viscous\ force}$

Viscous force, $F_v = \mu\left(\frac{\partial u}{\partial y}\right)A$

$$F_v \propto \mu.\frac{u}{L}.L^2 \propto \mu u L$$

Inertia force: momentum flow $= \dot{m}u$

$$\dot{m}u = \rho u A.A \propto \rho u^2 L^2$$

$$\Rightarrow R_e = \frac{\rho u^2 L^2}{\mu u L}, \quad \Rightarrow R_e = \frac{\rho u L}{\mu}$$

Location of transition point depends on

- Roughness of the surface, roughness hastens transition.
- Main stream flow. Turbulence in main stream hasten transition

- $$\mathrm{Re}_x = \frac{u_m x}{\upsilon}$$

$$\upsilon = \frac{viscosity, \mu}{density, \rho}$$

For $\mathrm{Re}_x < 10^5$, the laminar boundary layer is stable

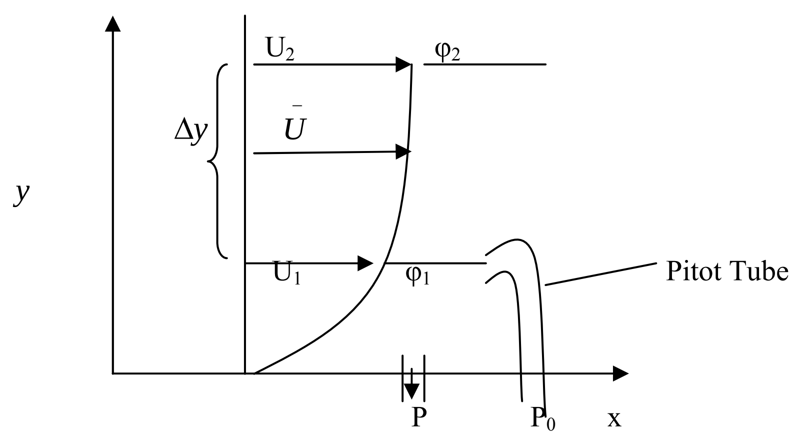
$\mathrm{Re}_x > 2 \times 10^6$, transition occurs even if surface is smooth and main stream is laminar.

- Pressure gradient

If $\frac{\partial P}{\partial x} \triangleright 0,\ \mathrm{Re}_{critical}\ is\ lowered$

If $\frac{\partial P}{\partial x} \triangleleft 0,\ \mathrm{Re}_{critical}\ is\ raised$

<u>Vorticity in the Boundary Layer</u>



$$P + \frac{1}{2}\rho V^2 = constant \quad = P_{st} + 0\ (when\ V = 0)$$

Variation of Pitot tubepressure across streamlines indicates vorticity

Bernoulli's Equation on φ_2 and φ_1

$$Po_1 = P + \frac{1}{2}\rho U_1^2 \ldots\ldots\ldots\ldots\ldots\ldots(1.2)$$

$$Po_2 = P + \frac{1}{2}\rho U_2^2 \ldots\ldots\ldots\ldots\ldots(1.3)$$

$$Po_2 - Po_1 = \frac{1}{2}\rho\left(U_2^2 - U_1^2\right) = \frac{1}{2}\rho\left(U_2 - U_1\right)\left(\frac{U_2 + U_1}{2}\right)\ldots\ldots\ldots\ldots(1.4)$$

$$\left(\frac{\Delta P}{\Delta y}\right) = \rho\bar{U}\frac{\Delta U}{\Delta y}\ldots\ldots\ldots\ldots(1.5)$$

$$\Omega_Z = \frac{1}{2}\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right)$$

In the boundary layer, $\frac{\partial V}{\partial x} \approx 0$

$$\omega_Z = 2\Omega_Z$$

$$= -\frac{\partial U}{\partial y}$$

$$= -\frac{1}{\rho\bar{U}}\left(\frac{\Delta P_O}{\Delta y}\right)\ldots\ldots\ldots\ldots(1.6)$$

In the limit as $\Delta y \to 0$

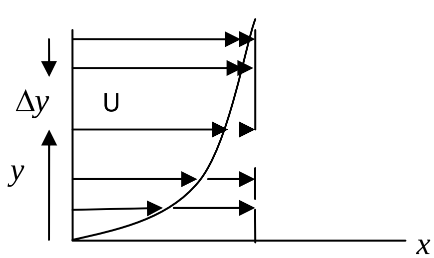
$$Vorticity = 2\Omega = \omega = -\frac{1}{\rho V}\frac{\partial P_O}{\partial y}$$

Therefore any variation of pressure in the boundary layer shows that the flow is rotational

<u>The Thickness of the boundary layer</u>

(i) Boundary layer thickness, δ is defined as the distance from solid surface at which the velocity reaches 99% of the main stream velocity.

(ii) Displacement thickness, δ*



Let U be the velocity at distance y.

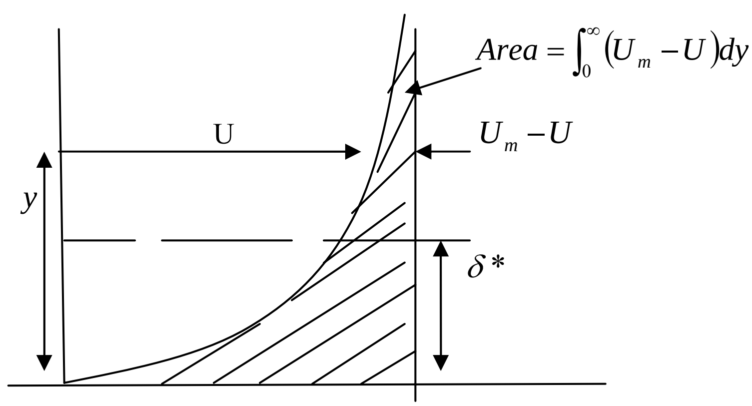
If there had been no boundary layer, the velocity would have been U_m. Reduction in flow due to the effect of the boundary layer $=(U_m - U)dy$

Total reduction in mass flow rate caused by the boundary layer $=\rho\int_0^\infty (U_m - U)dy$

Displacement thickness is defined by

$$\rho U_m \delta^* = \rho\int_0^\infty (U_m - U)dy$$

$$\delta^* = \frac{1}{U_m}\int_0^\infty (U_m - U)dy = \int_0^\infty \left(1-\frac{U}{U_m}\right)dy \qquad (1.7)$$



To reduce the total volume flow rate of a frictionless fluid by the same amount that the boundary layer does, the surface would have to be displaced outward by distance δ*. The concept of displacement thickness often allows us to consider the main flow as that of the frictionless fluid passing a displaced surface instead of an actual flow passing the actual surface.

(iii) Momentum thickness, θ

Fluid passing through elemental area $\delta y \times 1$ carries momentum at a rate $(\rho U \delta y)U$

In frictionless flow, the same mass would carry momentum flow rate $(\rho U \delta y)U_m$

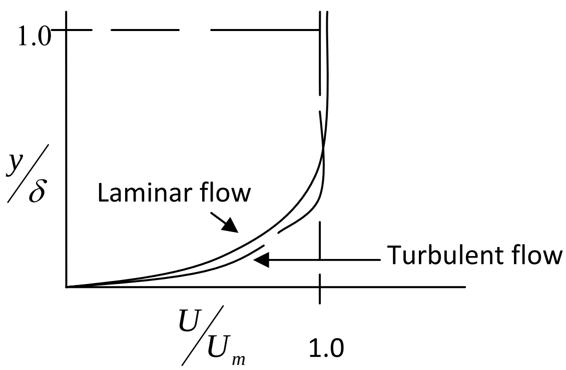
Total reduction in momentum flow rate due to boundary layer $=\int_0^w \rho(U_m - U)Udy$

Momentum thickness θ, is defined by

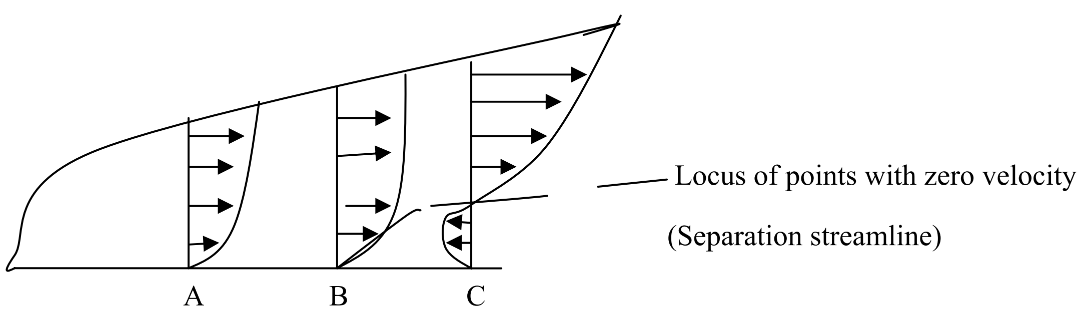
$$\left(\rho U_m \theta\right) U_m = \int_0^{\infty} \rho \left(U_m - U\right) U dy$$

$$\theta = \int_0^{\infty} \frac{U}{U_m}\left(1 - \frac{U}{U_m}\right) dy .... (1.8)$$

Velocity Profile and Shear Stress in the Boundary Layer



$$\partial P / \partial x \triangleright 0 \ \textit{adverse pressure gradient}$$



Point B = Separation point

$$\left.\frac{\partial U}{\partial y}\right|_{y=0} = 0 \ \ shear \ stress = 0$$

From point B, the flow is no longer able to follow the contour of the surface and breaks away from it. This phenomenon is termed separation.

Flow separation is caused by two factors

(i) Friction in the fluid (viscosity)

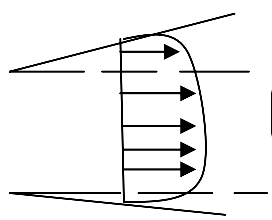
(ii) Adverse pressure gradient.

Separation streamline is the line of zero velocity dividing the forward and reversed flow which leave he surface at point B. Between the separation streamline and the wall are large irregular eddies as result of a reverse flow. In this region energy is dissipated as heat in the Eddies so pressure remains constant throughout the boundary B.

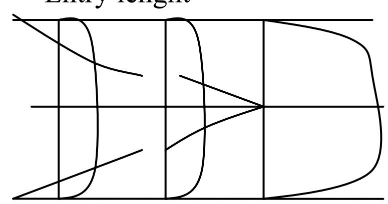
Laminar boundary layer is more susceptible to flow separation than the turbulent boundary layer since the velocity gradient near the wall in turbulent boundary layer is higher than that of the laminar boundary layer.

- The greater $\left(\frac{\partial P}{\partial x}\right)$, the sooner separation occurs.

- Boundary layer thickens rapidly in an adverse pressure gradient.

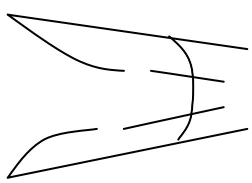


$\left(\frac{\partial P}{\partial x}\right) = 0$



Subsonic diffuser

Velocity

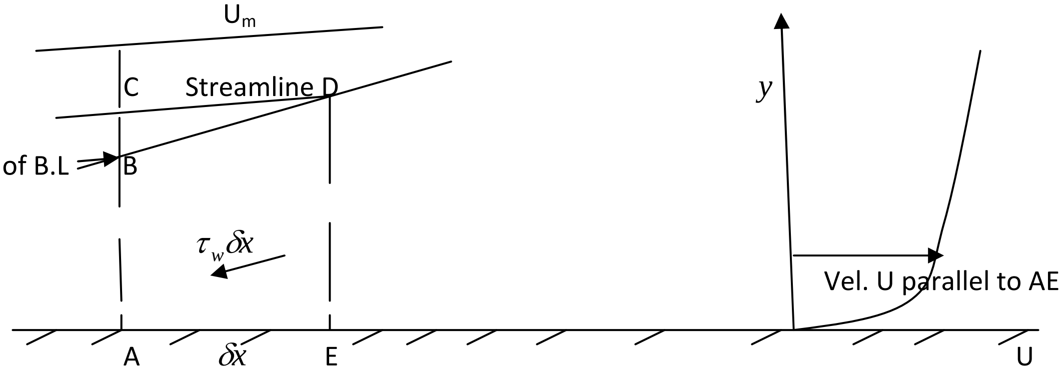


Subsonic nozzle $\left(\frac{\partial P}{\partial x}\right) \lhd 0$

Von Karman Momentum Integral Equation of the Boundary Layer

It is an approximate method based on momentum equation.

Consider a 2-D steady flow over a flat plate.



The boundary layer is of thickness δ, and its outer edge is represented by BD.

Let C be the point on AB produced which is on the same streamline as D

Consider ACDE as the control volume. Over surface AC the mean pressure is P

Over surface ED mean pressure $= P + \left(\frac{\partial P}{\partial x}\right)\delta x$

For a unit depth perpendicular plane, net force on control volume is

$$P.(AC) - \left[P + \left(\frac{\partial P}{\partial x}\right)\delta x\right]ED + \left(P + \tfrac{1}{2}\delta x\right)(ED - AC)$$

The expression reduces to

$$-\tfrac{1}{2}\left(\frac{\partial P}{\partial x}\right)\delta x(ED + AC)$$

$$As\ \delta x \to 0, AC \to ED\ and\ \text{expression}\ tends\ to -\left(\frac{\partial P}{\partial x}\right)\delta x(ED)$$

The total force on the control volume in x-direction

$$= -\tau_w \delta x - \left(\frac{\partial P}{\partial x}\right)\delta x(ED) \ldots\ldots\ldots\ldots(1.9)$$

The rate at which x-momentum is carried through AB $= \int_0^\delta \rho U^2 dy$

Rate at which x-momentum is carried through ED $= \int_0^\delta \rho U^2 dy + \frac{\partial}{\partial x}\left(\int_0^\delta \rho U^2 dy\right)\delta x$

Net rate of increase of x-momentum of the fluid passing through the control volume is

$$= \left[\int_0^\delta \rho U^2 dy + \frac{\partial}{\partial x}\left(\int_0^\delta \rho U^2 dy\right)\delta x\right] - \left[\int_0^\delta \rho U^2 dy + \rho U_m^2 (BC)\right]$$

$$= \frac{\partial}{\partial x}\left(\int_0^\delta \rho U^2 dy\right)\delta x - \rho U_m^2 (BC) \ldots\ldots\ldots\ldots\ldots(1.10)$$

Mass flow rate across AC = Mass flow rate across ED.

Mass flow rate across BC = $\rho U_m(BC)$ = Mass flow rate across ED – mass flow rate across AB

$$= \frac{\partial}{\partial x}\left(\int_0^\delta \rho U dy\right)\delta x$$

$$\therefore \rho U_m^2 (BC) = U_m \frac{\partial}{\partial x}\left(\int_0^\delta \rho U dy\right)\delta x \ldots\ldots\ldots\ldots\ldots(1.11)$$

Substituting (1.11) in (1.10)

Rate of increase in x-momentum of fluid passing through the control volume $= \frac{\partial}{\partial x}\left(\int_0^\delta \rho U^2 dy\right)\delta x - U_m \frac{\partial}{\partial x}\left(\int_0^\delta \rho U dy\right)\delta x \ldots\ldots\ldots\ldots(1.12)$

Equating this to the total x-force on the control volume and dividing through by ∂x,

$$\tau_w + \frac{\partial P}{\partial x}(ED) = U_m \frac{\partial}{\partial x}\int_0^\delta \rho U dy - \frac{\partial}{\partial x}\int_0^\delta \rho U^2 dy \ldots\ldots\ldots\ldots(1.13)$$

$\frac{\partial P}{\partial y} \approx 0$ In the boundary layer since fluid particle in y-direction is negligible.

Outside the boundary layer, it is approximately flow of an ideal fluid.

$$P + \tfrac{1}{2}\rho U_m^2 = constant$$

$$\frac{\partial P}{\partial x} + \rho U_m \frac{\partial U_m}{\partial x} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1.14)$$

Substituting for (1.14) in (1.13) and noting that

$$ED = \delta = \int_0^\delta dy \quad and \quad \rho = constant$$

$$\Rightarrow \tau_w - \rho U_m \frac{\partial U_m}{\partial x}\int_0^\delta dy = \rho U_m \frac{\partial}{\partial x}\int_0^\delta U dy - \rho \frac{\partial}{\partial x}\int_0^\delta U^2 dy ....(1.15)$$

$$Since\ U_m \frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(U_m U) - U\frac{\partial U_m}{\partial x}$$

Equation (1.15) becomes

$$\tau_w = \rho \frac{\partial}{\partial x}\int_0^\delta U_m U dy - \rho \frac{\partial U_m}{\partial x}\int_0^\delta U dy - \rho \frac{\partial}{\partial x}\int_0^\delta U^2 dy + \rho \frac{\partial U_m}{\partial x}\int_0^\delta U_m dy ....(1.16)$$

$$\tau_w = \rho \frac{\partial}{\partial x}\int_0^\delta (U_m - U) U dy + \rho \frac{\partial U_m}{\partial x}\int_0^\delta (U_m - U) dy ....(1.16a)$$

Since $(U_m - U)$ becomes zero at the edge of the boundary layer, the upper limit of both integrals may be changed to ∞. Then from the definition of the displacement and momentum thickness, equation (1.16) simplifies to

$$\tau_w = \rho \frac{\partial}{\partial x}(U_m^2 \theta) + \rho \frac{\partial U_m}{\partial x} U_m \delta^* ....(1.17)$$

Equation (1.17) is the momentum equation for the boundary layer.

Special case, if

$$\frac{dP}{dx} = 0, equation\ (1.14)\ shows\ that$$

If $\frac{\partial U_m}{\partial x} = 0\ and\ equation\ (1.17)\ reduces\ to$

$$\frac{\tau_w}{\rho U_m^2} = \frac{\partial \theta}{\partial x} ....(1.18)$$

Laminar Boundary Layer on a smooth flat plate with zero pressure gradient

At low

$$\mathrm{Re} = \frac{\rho UL}{\mu}$$

There are flows in which the laminar boundary is important in all laminar flow

$$\tau_w = \mu \frac{\partial u}{\partial y} \; and \;\; \tau_w = \mu \frac{\partial u}{\partial y} \updownarrow_{y=0}$$

Assuming

$$\frac{\partial P}{\partial x} = 0 \Rightarrow \frac{\partial U_m}{\partial x} = 0$$

Substituting for

$$\tau_w \; and \; \frac{\partial U_m}{\partial x} = 0 \; in \; equation \, (1.16)$$

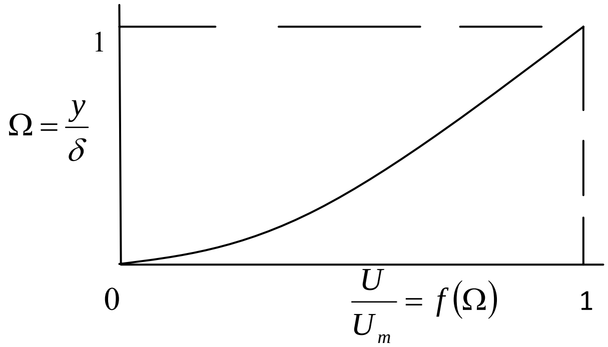
$$\Rightarrow \; \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \rho \frac{\partial}{\partial x} \int_0^\delta (U_m - U) U dy ....(1.19)$$

We need velocity profile in the boundary layer.

Assume that u increase from y=0 to U = U_m at y=δ at any value x

$$\Omega = \frac{y}{\delta} \Rightarrow \Omega = 0 \;\; at \;\; y = 0 \; , \; \Omega = 1 \;\; at \;\; y = \delta$$

Then U/U_m is always the same function of Ω at any value of x.



$$Subtituting\ y = \Omega\delta\ and\ U = U_m f(\delta)\ in\ equ(1.19)$$

$$\Rightarrow \frac{\mu}{\delta} U_m \left[\frac{\partial f(\Omega)}{\partial \Omega}\right]_{\Omega=0} = \rho \frac{\partial}{\partial x}\left[U_m^2 \delta \int_0^1 (1 - f(\Omega)) f(\Omega) d\Omega\right]$$

Since f (Ω) is assumed independent of x

$$\Rightarrow \int_0^1 (1 - f(\Omega) f(\Omega) dy)\ may\ be\ written\ as\ a\ constant\ A\ and\ B = \left[\frac{\partial f(\Omega)}{\partial \Omega}\right]_{\Omega=0}$$

$$The\ equation\ may\ be\ re-written\ as$$

$$\frac{\mu}{\delta} U_m B = \rho \frac{\partial}{\partial x}\left(U_m^2 A \delta\right) = \rho U_m^2 A \frac{d\delta}{dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.20)$$

Multiplying by δ/U_m and integrating with respect to x

$$\Rightarrow \mu B x = \rho U_m A \delta^2/2 + constant \ldots\ldots\ldots\ldots\ldots (1.21)$$

If x is measured from leading edge of plate, δ=0 when x=0, so the constant in equation (1.21) is equal to zero.

$$\therefore \delta = \left(\frac{2\mu B x}{\rho U_m A}\right)^{1/2} = \left(\frac{2B}{A}\right)^{1/2} \frac{x}{\mathrm{Re}_x^{1/2}} \ldots\ldots\ldots\ldots\ldots\ldots (1.22)$$

$$Where\ \mathrm{Re}_x = \frac{\rho U_m x}{\mu},\ i.e\ local\ Reynold's\ number$$

From equations (1.20) and (1.22)

$$\tau_w = \rho U_m^2 A \frac{d\delta}{dx} = \rho U_m^2 A \left(\frac{2\mu B}{\rho U_m A} \right)^{1/2} \tfrac{1}{2} x^{-1/2} = \rho U_m^2 \left(\frac{AB}{2\,\text{Re}_x} \right)^{1/2} \text{.............}(1.23)$$

Total friction force between x=0 and x=L for unit width on one side of the plate is

$$F = \int_0^l \tau_w dx = \left[\rho U_m^2 A \delta \right]_0^l$$

$$F = \left(2AB\mu\rho U_m^3 L \right)^{1/2} \text{....................}(1.24)$$

The Dimensionless skin friction coefficient

$$C_F = \frac{Mean\ friction\ stress}{\tfrac{1}{2}\rho U_m^2}$$

$$= \frac{F/(L \times I)}{\tfrac{1}{2}\rho U_m^2} = \frac{F}{\tfrac{1}{2}\rho U_m^2 L}$$

$$C_F = 2\left(\frac{2AB\mu}{\rho U_m L} \right)^{1/2} = 2\frac{(2AB)^{1/2}}{\text{Re}^{1/2}} \text{...............}(1.25)$$

Equation (1.22) shows that the laminar boundary layer

$$\delta \propto x^{1/2} \ and\ \delta \propto \frac{1}{U_m^{1/2}}$$

Equations (1.23) and (1.24) shows that

$$\tau_w \propto \frac{1}{\text{Re}_x^{1/2}}$$

And the total drag force on the plate due to friction is

$$F \propto U_m^{3/2} \ and\ F \propto L^{1/2}$$

To evaluate δ from equation (1.2) we need to calculate A and B. Hence the form of the function f (Ω) is required.

<u>Simplest Assumption</u>

Since U varies from 0 at wall to U_m at y=δ, assume that τ decreases linearly from τ_w to 0 at y=δ

$$\tau = k(\delta - y) \ldots\ldots\ldots\ldots\ldots (1.26)$$

$$Setting\ \tau = \mu \frac{\partial U}{\partial x}\ and\ \text{int}egrating\ \mu U = k\left(y\delta - {y^2}/{2} \right) + c$$

$$U = 0\ \ when\ y = 0 \Rightarrow c = 0$$

$$Putting\ y = \Omega\delta\ and\ dividing\ by\ \mu U_m$$

$$\frac{U}{U_m} = \frac{k\delta^2}{\mu U_m}\left(\Omega - {\Omega^2}/{2}\right)$$

$$when\ \Omega = 1,\ \frac{U}{U_m} = \frac{k\delta^2}{\mu U_m}.\frac{1}{2}$$

$$\therefore \frac{k\delta^2}{\mu U_m} = 2\ \ for\ U = U_m$$

$$\therefore\ f(\Omega) = 2\Omega - \Omega^2 \ldots\ldots\ldots\ldots\ldots (1.27)$$

$$A = \int_0^1 (1 - f(\Omega)) f(\Omega) d\Omega$$

$$= \int_0^1 \left(1 - 2\Omega + \Omega^2\right)\left(2\Omega - \Omega^2\right)$$

$$= \int_0^1 \left(2\Omega - 5\Omega^2 + 4\Omega^3 - \Omega^4\right) d\Omega$$

$$= \left[\Omega^2 - {5\Omega^3}/{3} + \Omega^4 - {\Omega^5}/{5}\right]_0^1$$

$$A = \frac{2}{15}$$

$$B = \frac{\partial f(\Omega)}{\partial \Omega}\Big|_{\Omega=0} = \frac{\partial}{\partial y}\left(2\Omega - \Omega^2\right)\Big|_{\Omega=0}$$

$$= 2 - 2\Omega\big|_{\Omega=0}$$

$$\therefore B = 2$$

$$\delta = \left(\frac{2B}{A}\right)^{1/2} \frac{x}{\mathrm{Re}_x^{1/2}}$$

$$= \left(\frac{4\times 15}{2}\right)^{1/2} \cdot \frac{x}{\mathrm{Re}_x^{1/2}}$$

$$\delta = \frac{5.48x}{\mathrm{Re}_x^{1/2}} \ldots\ldots\ldots\ldots\ldots(1.28)$$

$$\delta^* = \int_0^{\delta}\left(1-\frac{U}{U_m}\right)dy$$

$$= \delta\int_0^1 (1-f(\Omega))d\Omega$$

$$= \delta\int_0^1 (1-2\Omega+\Omega^2)d\Omega$$

$$\delta = \frac{\delta}{3}$$

$$\delta = \frac{1}{3}\cdot\frac{5.48x}{\mathrm{Re}_x^{1/2}} \ldots\ldots\ldots\ldots\ldots\ldots(1.29)$$

Momentum Thickness

$$\theta = \delta\int_0^1 f(\Omega)(1-f(\Omega)d\Omega)d\Omega = A\delta$$

$$= \frac{2}{15}\times\frac{5.48x}{\mathrm{Re}_x^{1/2}}$$

$$\theta = \frac{0.730x}{\mathrm{Re}_x^{1/2}} \ldots\ldots\ldots\ldots\ldots\ldots(1.30)$$

From equation (1.23)

$$\tau_w = \rho U_m^2\left(\frac{AB}{2\,\mathrm{Re}}\right)^{1/2}$$

$$= 0.365\frac{\rho U_m^2}{\mathrm{Re}_x^{1/2}} \ldots\ldots\ldots\ldots\ldots\ldots(1.31)$$

There are other assumptions about f (Ω), e.g.

$$(i)\ f(\Omega) = 2\Omega - \Omega^2$$

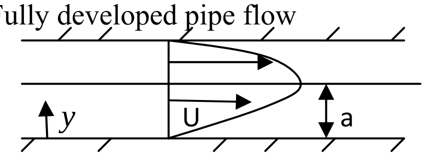
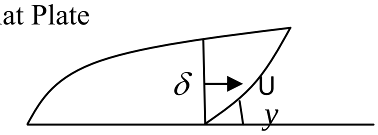
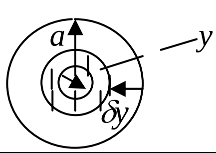
$$(ii)\ f(\Omega) = \frac{3}{2}\Omega - \frac{1}{2}\Omega^2$$

$$(iii)\ f(\Omega) = 2\Omega - 2\Omega^3 + \Omega^4$$

$$(iv)\ f(\Omega) = \sin\left(\frac{\pi\Omega}{2}\right)$$

Turbulent boundary layer on a smooth flat plate with zero pressure gradients

Most of the boundary layer encountered in practice is turbulent for most of their length. Much experimental information is available about turbulent flow in circular pipes and Prandtl suggested that this could be used to study boundary layer on flat plate on the ground that the boundary layers in the two cases are not essentially different.

Fully developed pipe flow  1. Velocity profile from extensive experimental data $\frac{U}{U_{max}} = \left(\frac{y}{a}\right)^{1/2}$	Flat Plate  Prandtl suggested $V_{max} = U_m$ $a = \delta$ $\Rightarrow \frac{U}{U_m} = \left(\frac{y}{\delta}\right)^{1/n}$ ....(1.32) *where n = 7*
2. Wall shear stress Prandtl considered a hydraulically smooth pipe. $C_F = \frac{\tau_w}{\frac{1}{2}\rho \bar{U}^2} = \frac{0.079}{\left(\frac{\rho \bar{U} d}{\mu}\right)^{1/4}}$ *Blasins equation where d = 2a* $\bar{U} = \frac{Q}{A} = mean\ velocity$ $\bar{U} = \frac{1}{\pi a^2}\int_0^a U 2\pi(a-y)dy$ $= 0.817U_{max}$ $\delta A = 2\pi(a-y)dy$ $\tau_w = 0.2773\rho U_m^2 \left(\frac{\upsilon}{Ud}\right)^{1/4}$ ....(1.34)  	*Wall shell stress* $d = 25$ $\bar{U} = 0.817U_\infty$ ....(1.33) Prandtl assumed that Blasins equation is valid if we make these changes $\tau_w = 0.0225\rho U_\infty^2 \times \left(\frac{\mu}{\rho U_\infty \delta}\right)^{1/4}$ ....(1.35)

Notes on the table

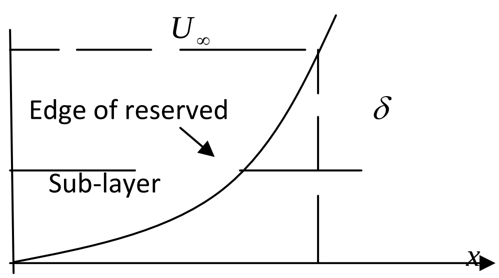
(i) Equation (1.32) is applicable for moderate Reynolds number only, i.e.

$$\frac{U_m x}{\upsilon} \lhd 10^7$$

(ii) The equation is not applicable at the wall since

$$\frac{\partial U}{\partial y} = \tfrac{1}{7} U_\infty \delta^{-\frac{1}{7}} y^{-\frac{6}{7}} = \infty \ at \ y = 0$$

The viscous sub layer is adjacent to the wall. Assume its velocity to be linear and tangency to the seventh root $\left(\sqrt[7]{\ }\right)$ profile at the point where the viscous merges with the turbulent part of the boundary layer.



For the case under consideration

$$\frac{\partial P}{dx} = 0 \Rightarrow \frac{\partial U_\infty}{\partial x} = 0$$

Substituting equations (1.34) and (1.32) in (1.16a)

$$0.0225 \rho U_\infty^2 \left(\frac{\upsilon}{U\infty\delta} \right)^{\frac{1}{4}} = \rho \frac{\partial}{\partial x} \int_0^\delta U_\infty^2 \left(1 - \left(\frac{y}{\delta} \right)^{\frac{1}{7}} \left(\frac{y}{\delta} \right)^{\frac{1}{7}} \right) dy$$

$$= \rho \frac{\partial}{\partial x} \left(\frac{7}{72} U_\infty^2 \delta \right)$$

$$= \frac{7}{72} \rho U_\infty^2 \frac{\partial \delta}{\partial x} \dots\dots\dots\dots (1.36)$$

Integrating from zero to δ in equation (1.37) is justified since the thickness of the viscous sub layer is smaller. Rearrange (1.36) to get

$$\delta^{1/4} d\delta \equiv 0.231\left(\frac{\upsilon}{U_w}\right)^{1/4} dx + constant \ldots\ldots\ldots\ldots\ldots(1.38)$$

Total derivative has replaced partial since δ is a function of x only.

Integrating

$$\frac{4}{5}\delta^{5/4} = 0.231\left(\frac{\upsilon}{U\infty}\right)^{1/4} x + constant$$

Assuming that the laminar portion of the boundary layer is small, Prandtl showed that reasonably good results are obtained if the boundary layer is assumed to be turbulent from x=0 (leading edge).

This makes the constant in equation (1.38) to be zero and

$$\frac{\delta}{x} = 0.370\,\mathrm{Re}_x^{-1/5} \ldots\ldots\ldots\ldots\ldots(1.39)$$

Momentum and displacement thickness:

$$\theta = \int_0^{\delta}\left(1-\frac{U}{U_\infty}\right)\frac{U}{U_\infty}dy$$

Substituting in the assumed velocity profile

$$\theta = \int_0^{\delta}\left[\left(\frac{y}{\delta}\right)^{1/7} - \left(\frac{y}{\delta}\right)^{2/7}\right]dy$$

$$Let\frac{y}{\delta} = \Omega, \quad \therefore dy = \delta d\Omega$$

$$\theta = \delta\int_0^{1}\left(\Omega^{1/7} - \Omega^{2/7}\right)d\Omega$$

$$\theta = \delta\left(\frac{7}{8}\Omega^{8/7} - \frac{7}{9}\Omega^{9/7}\right)_0^1$$

$$\theta = \frac{7}{72}\delta \ldots\ldots\ldots\ldots\ldots\ldots(1.40)$$

Similarly, the displacement thickness may be shown to be

$$\delta^* = \frac{\delta}{8} \ldots\ldots\ldots\ldots\ldots\ldots (1.41)$$

$$\frac{\delta^*}{x} = 0.0463 \mathrm{Re}_x^{-1/5}$$

Total drag force F on one side of the plate per unit width $F = \int_0^1 \tau_w dx$

Substituting from (1.35) and (1.39) in this equation

$$F = \int 0.0225 \rho U_\infty^2 \left(\frac{\upsilon}{U_\infty} \right)^{1/4} \times 0.370^{-1/4} x^{-1/4} . \left(\frac{U_\infty x}{\upsilon} \right)^{1/2} dx$$

$$\approx 0.0360 \rho U_\infty^2 \left(\frac{\upsilon}{U_\infty L} \right)^{1/5} L$$

The overall skin-friction coefficient is

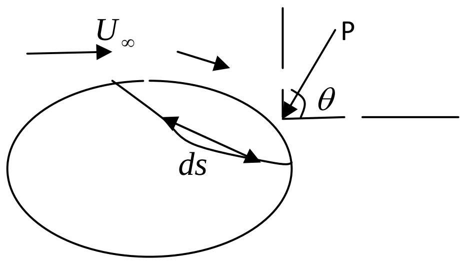
$$C_F = \frac{mean\ friction}{1/2 \rho U_\infty^2} = \frac{F/L}{1/2 \rho U_\infty^2} = 0.072 (\mathrm{Re}_x)^{-1/5} \ldots\ldots\ldots\ldots\ldots (1.42)$$

Measurement of drag force indicate that the value is move nearly

$$C_F = 0.074 (\mathrm{Re}_L)^{-1/5} \ldots\ldots\ldots\ldots (1.43)$$

$$(\mathrm{Re}_x) = 5 \times 10^5 \ to \ 10^7$$

Lift and Drag on a Body



$$Skin\ \ friction\ drag\ , D_f\ = \oint \tau_w \sin\theta ds ....(1.44)$$

$$\tau_w = \mu\left(\frac{\partial U}{\partial y}\right)_{y=0} \quad for\ \ flat\ \ plate$$

$$\Pr essure\ drag\ \ D_p\ = \oint P\cos\theta\, ds = \ form\ drag\ ....(1.45)$$

Profile Drag = Skin friction drag + Pressure drag = $D_F + D_P$

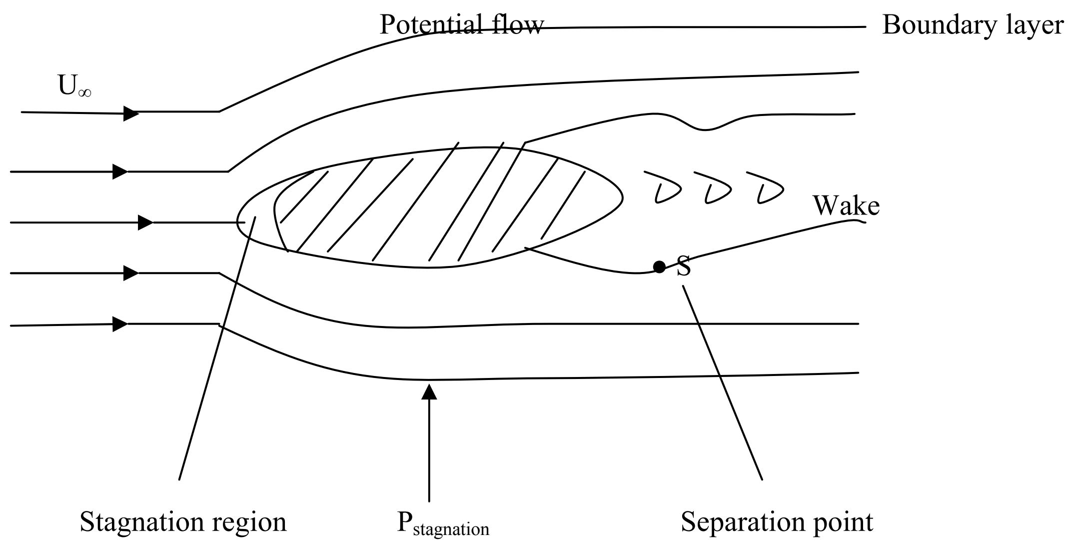
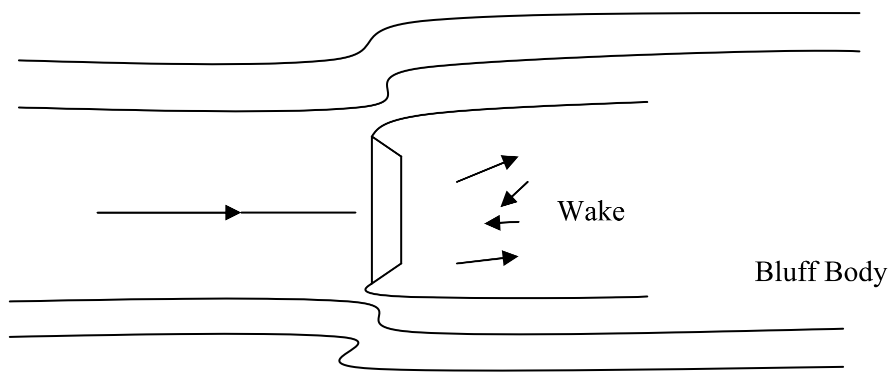


Fig: Flow regimes around an immersed streamlined body

$$P + \frac{1}{2}\rho_{\downarrow=0}U^2 = constant$$



Wake = Region of eddying motion downstream of the separation point.

For streamlined body, skin friction makes the major contribution to the total drag.

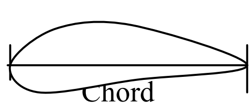
For Bluff Body, Pressure Drag >> Skin Friction Drag.

$$Drag\ coefficient,\ C_D = \frac{Total\ drag\ force}{\frac{1}{2}\rho U_\infty^2 A}$$

A = Projected Area Perpendicular to oncoming stream

<u>For Airofoils</u>

A = Product of Span and mean chord



<u>For a Flat plate</u>

A = Area of both sides of the plate.

Similar summation as in equation (1.44) and (1.45) can be done for the force components perpendicular to U_∞ to give the lift.

$$L = \frac{1}{2}C_L \rho U_\infty^2 A..............(1.46)$$

Where C_L = Lift coefficient.

Resultant Force on the body

$$F = \sqrt{L^2 + D^2}$$

$$F = \tfrac{1}{2}\rho U_\infty^2 A\sqrt{C_L^2 + C_D^2} \;..........(1.47)$$

Examples 1.

Calculate the laminary boundary layer and the displacement thickness at a distance of 2.5m from the leading edge of a train moving at 6m/s. Take the kinematic viscosity of air as $1.55\text{x}10^{-5}$m/s?

Solution

$$\text{Re}_x = \frac{Ux}{\upsilon}$$

$$= \frac{6 \times 2.5}{1.55 \times 10^{-5}}$$

$$\text{Re}_x = 9.68 \times 10^5$$

$$\textit{From the relation}\ \frac{\delta}{x} = \frac{4.96}{\text{Re}_x^{1/2}}$$

$$\Rightarrow \delta = \frac{4.96 \times 2.5}{\sqrt{9.68 \times 10^5}}$$

$$\delta = 12.60mm$$

$$\frac{\delta^*}{x} = 1.73\text{Re}_x^{-1/2}$$

$$\delta^* = \frac{1.73x}{\sqrt{\text{Re}_x}}$$

$$\delta^* = \frac{1.73 \times 2.5}{\sqrt{9.68 \times 10^5}}$$

=0.00440

=4.40mm

Example 2

In example 1, consider that the free-stream turbine is such that transition takes place at a Reynolds number predicted by Hansen, namely at 3.2x10^5. Compute the boundary-layer thickness at transition for laminar boundary layer and compare it to the boundary-layer thickness computed from turbulent flow at the same position. Next, find the boundary-layer thickness at the leading position of the turbo train?

Solution

$$\mathrm{Re}_{critical} = 3.2\times10^5$$

$$\mathrm{Re} = \frac{Ux_T}{\upsilon} = \mathrm{Re}_{critical}$$

$$x_T = \frac{\upsilon.\mathrm{Re}_{critical}}{U}$$

$$= \frac{1.55\times10^{-5}\times3.2\times10^5}{6}$$

$$= 0.827m$$

Boundary layer thickness, δ for laminar b.l.

$$\frac{\delta}{x} = \frac{4.96}{\sqrt{\mathrm{Re}_x}}$$

$$\delta = \frac{4.96\times0.827}{\sqrt{3.2\times10^5}}$$

$$= 7.25mm$$

Boundary layer thickness, δ for turbulent b.l

$$\frac{\delta}{x} = \frac{0.37}{\mathrm{Re}_x^{1/5}}$$

$$\delta = \frac{0.37x}{\mathrm{Re}_x^{1/5}}$$

$$= \frac{0.37\times0.827}{\left(3.2\times10^5\right)^{1/5}}$$

$$= 24.25mm$$

$Boundary\ layer\ thickness, \delta\ at\ leading\ edge$

$$\delta = \frac{0.37x}{\mathrm{Re}_x^{1/5}}$$

$$x = 2.5m$$

$$\Rightarrow \delta = \frac{0.37 \times 2.5}{\mathrm{Re}_x^{1/5}}$$

$$\mathrm{Re}_x = \frac{Ux}{\upsilon} = \frac{6 \times 2.5}{1.55 \times 10^{-5}} = 967{,}741.94$$

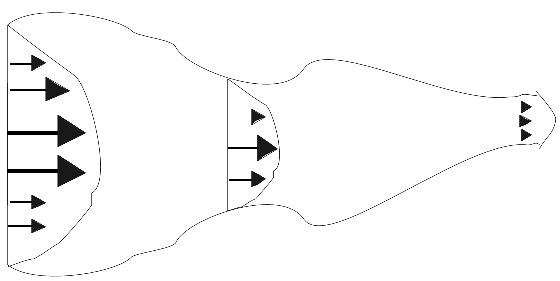
$$\delta = \frac{0.37 \times 2.5}{(967741.94)^{1/5}} = 0.05875$$

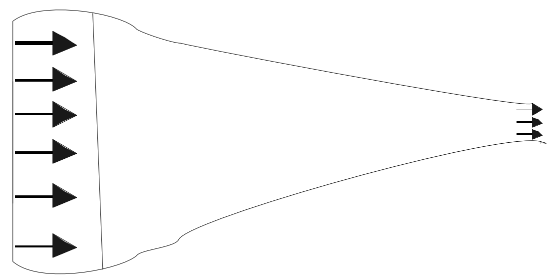
$$\delta = 58.75mm$$

CHAPTER 1: PRESSURE WAVES IN FLUIDS

1. ONE DIMENSIONAL FLOW

Strictly speaking, we need 3-D analysis to find complete solution to compressible flow equations. But there are many flow problems that can be solved to a good-engineering approximation with the use of 1-D analysis.





Velocity profile in real flow ***Velocity profile in 1-D Flow***

1-D flow- flow variables are functions of one

Space coordinates i.e. $\frac{\partial}{\partial y} + \frac{\partial}{\partial t} = 0$

→ Velocity is uniform across cross-sectional area.

→ No velocity components in y or t direction.

In true 1-D flow, area changes are not allowed. The more gradual the area changes, the better the 1-D approximation. When area changes are gradual, we talk of quasi one-dimensional.

<u>Continuity equation:</u>

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

For 1-D flow

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\rho \partial u}{\partial x} + \frac{u \partial \rho}{\partial x} = 0$$

<u>For steady flow</u>

$$\frac{\rho \partial u}{\partial x} + \frac{u \partial \rho}{\partial x} = 0$$

In gas dynamics, viscosity is normally neglected. Entering equation for 1-D flow is

$$\frac{\partial u}{rt} + \frac{U \partial u}{rx} = \frac{-\partial P}{\rho \partial x} + f_x$$

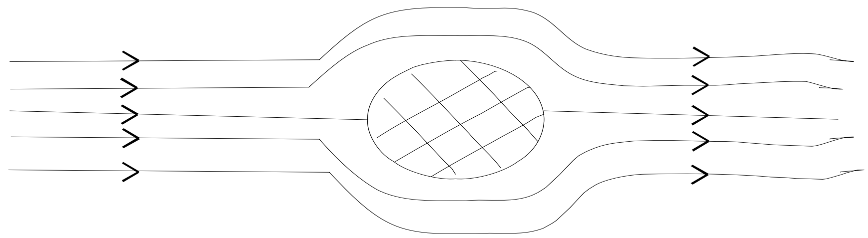
<u>For steady flow</u>

$$\frac{U \partial u}{\partial x} = -\frac{1 \partial P}{\rho \partial x} + f_x$$

For 1-D flow without body force, entering equation becomes:

$$\frac{U \partial u}{\partial x} = -\frac{1 \partial P}{\rho \partial x}$$

1.2 <u>WAVE PROPAGATION IN COMPRESSIBLE MEDIA</u>

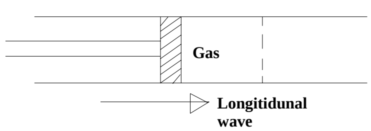


Streamline pattern obtained for incompressible flow over a circular cylinder.

The fluid particles are able to sense the presence of the body before reaching it. This suggests the existence of a signal mechanism whereby a fluid particle can be forewarned of the disturbance in the flow ahead of it.

Velocity of signal waves sent from the body relative to the moving fluid is greater than the absolute fluid velocity.

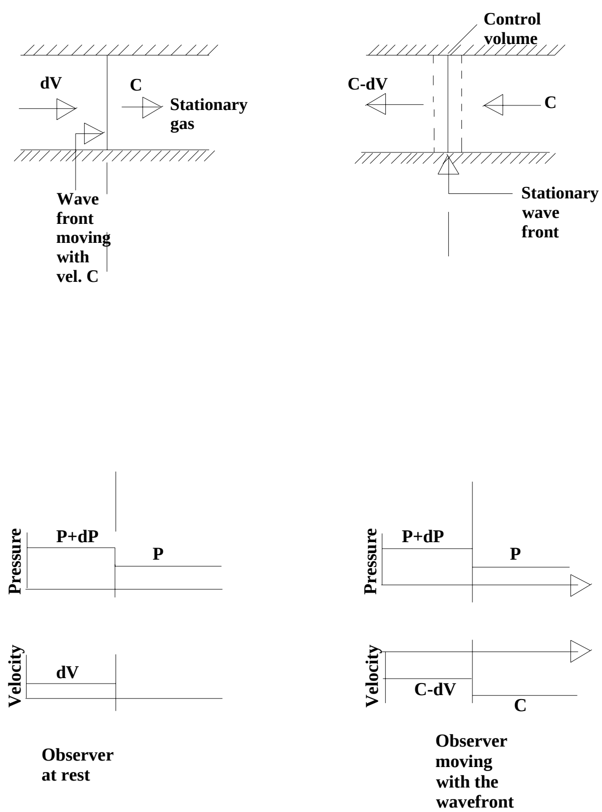
If fluid particles were to move faster than the signal waves, the fluid will not be able to sense the body before actually reaching it, and very abrupt changes in velocity vector and others would occur



If piston is given a sudden push to the right, in the first instant, a layer of fluid piles up in front of the piston and is compressed, the remainder of the gas will be unaffected. The compression waves created by the piston then moves through the gas until all the gas is able to sense the movement of the piston. If the impulse given to the piston is infinitesimally small, the wave is called a sound wave and the resultant compression wave move through the gas at a velocity equal to the speed of sound. For incompressible medium, no change in density is possible. The velocity wave propagation is infinite. The higher the compressibility, the lower will be the velocity of sound in that substance.

1.3 VELOCITY OF A PLANE PRESSURE PULSE

Consider an infinitesimal pressure wave proceeding along a pipe of uniform cross-section area.



Consider the control volume shown. Shear forces on the control volume at the wall is negligible compared with pressure forces.

Momentum equation

$$Force = \dot{m}\, dV$$

$$A[\rho-(P+dP)]=\dot{m}[(C-dV)-C\ ]....................(1.1)$$

Where A is the cross sectional area

Simplifying and noting from continuity equation that: $\dot{m}=\rho AC\ ....(1.2)$

We'll get $dP=\rho CdV\ ....(1.3)$

$$dV=decrease\ in\, velocity\, of\ motion$$

Continuity equation written for the fluid on both sides of the wave front

$$\rho CA=(\rho+d\rho)(C-dV)A\ ....(1.4)$$

Equation (1.4) reduces to

$$\frac{d\rho}{\rho}=\frac{dV}{C}....................(1.5)$$

Combining (1.5) and (1.3) we get

$$C^2=\left(\partial P/\partial\rho\right)_S\ \ or\ C=\sqrt{\left(\partial P/\partial\rho\right)_S}\ \ ....(1.6)$$

The ratio ∂P/∂ρ is written as in partial derivatives at constant entropy because the variation in p and T are vanishingly small and so the process is nearly reversibly. The comparative rapidity of the process and the smallness of the temperature of the variation makes the process nearly adiabatic.

For a perfect gas

$$\frac{P}{\rho^{r}} = Cons\tan t \ldots\ldots\ldots\ldots..(1.7)\ and$$
$$P = \rho RT \ldots\ldots\ldots\ldots..(1.8)$$

Putting (1.7) into log form

$$\Rightarrow \ln P = r\ln\rho = cons\tan t \ldots\ldots\ldots\ldots(1.9)$$

Differentiating

$$\frac{dP}{\rho} = \frac{rdP}{\rho}$$
$$\left(\frac{\partial P}{\rho}\right)_{S} = \frac{\partial P}{\rho} = \partial RT \ldots\ldots\ldots\ldots\ldots..(1.10)$$

Velocity of sound in a perfect gas:

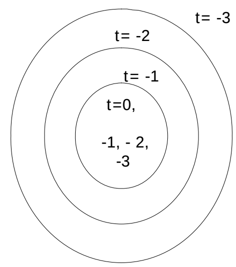
$$C^{2} = \frac{rP}{\rho} = RrT = \frac{rR_{U}T}{W} \ldots\ldots\ldots\ldots\ldots..(1.11)$$

Where R_u= universal gas constant, W = molar weight.

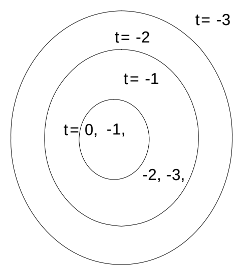
1.4 <u>Pressure field created by a moving point</u>

Consider the pressure field by a point source of disturbance moving at uniform linear speed through a compressible medium.

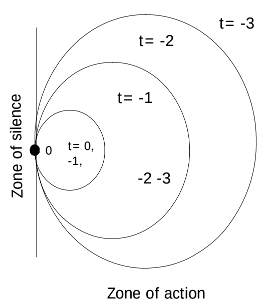
(a) Incompressible flow, V/C =0



(b) Subsonic flow V/C =1/2



(c) Sonic flow V/C =1



(d) Supersonic motion, V/C >1

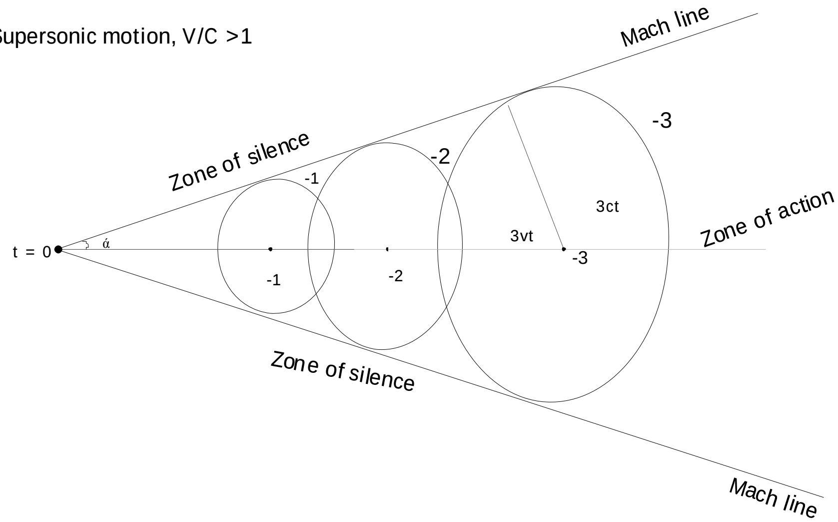


Fig Pressure pulse patterns compared for different values of the speed of the source compared with the speed of the sound in the fluid.

Incompressible flow

When the medium is incompressible or when the speed of the moving point disturbance is small compared with the speed of it sound, the pressure pulse spread uniformly in all direction.

Subsonic flow

The pressure disturbance is felt in all direction and at all point in space (Negatively dissipation due to viscosity), but the pressure per time is asymmetrical.

Supersonic flow

All pressure disturbances are included in a curve which has the point source at its apex, and the effect of the disturbance is not felt upstream of the point source. The curve within which the disturbances are confirmed is called the mach come. For the case of the sonic flow v/c=1 and the mach come is a plane.

Karman's rules of supersonic flow

The rules apply exactly only for small disturbances, but are usually qualitatively applicable for large disturbances.

Rule of forbidden signals

The effect of pressure changes produce by a body moving at a speed faster than sound cannot reach point ahead of the body.

The zone of silence and the zone of action

A stationary source in a supersonic speed produces effect only on points that is on or inside the mach curve extending downstream the mach come.

The rule of concentrated action

The pressure disturbance is largely concentrated in a neighborhood of the mach curve that forms the outer limit of the zone of action. These rules explain why a projective moving at a supersonic speed cannot be heard until the wave attached to the nose of the body passes over the ear of the observer; and why the latter does occur, the noise is concentrated in a crack called sonic boom or sonic bang.

The mach number and mach angle

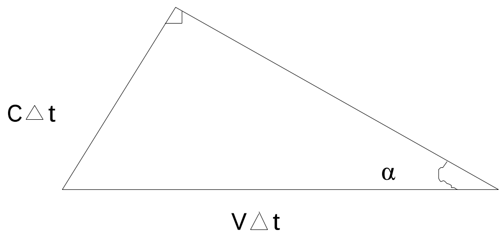
$$M = \frac{V}{C},\ M^2 = \frac{V^2}{C^2} = \frac{V^2}{rRT} = \frac{Directed\ K.E}{Random\ K.E}$$

$$= \frac{Inertia\ Force}{\Pr essure\ Forces\ due\ to\ compressibility}$$

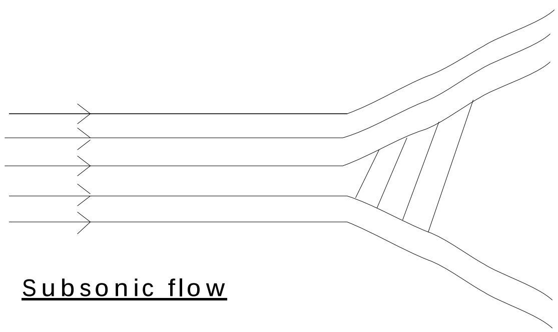
$$C^2 = rRT = \frac{rP}{\rho}$$

$$C = Ther\mathrm{mod}\,ynamic\ property,\ C^2 = \left(\frac{\partial P}{\partial \rho}\right)_S$$

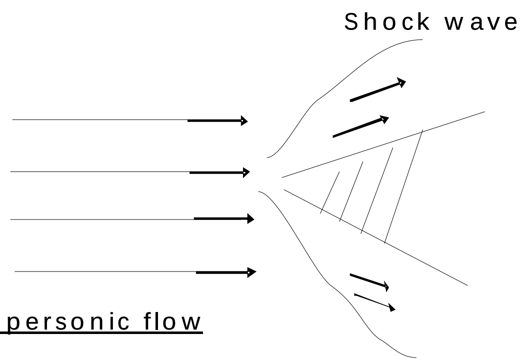
It varies from point to point



$\sin \alpha = C/V = 1/M$



Subsonic flow



Supersonic flow

On like the point projectile discussed previously, the body now present a finite disturbance to the flow. The wave pattern obtained is a result of the addition of the individual mach waves emitted for each point on the wave.This non-linear addition yields a compression shock wave across which causes finite changes in velocity pressure and other flow properties.

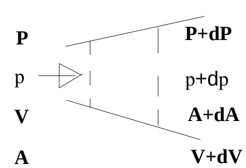
CHAPTER 2: ISENTROPIC FLOW OF A PERFECT GAS

We shall concern ourselves with compressible, isentropic flow through a varying area channels, such as nozzles, diffusers, and turbine-blade passages

Assumptions

The flow is 1-D, steady, friction and heat transfer are negligible and change in pontetial energy and gravitatimal forces are neglected.

Equation of motion



Continuity equation for steady in integral form

$$\int_C \int_S \rho \vec{V} \bullet d\vec{A} = 0 \ldots\ldots\ldots\ldots(2.1)$$

For this case

$$(\rho + d\rho)(A + dA)(V + dV) - \rho AV = 0 \ldots\ldots\ldots\ldots(2.2)$$

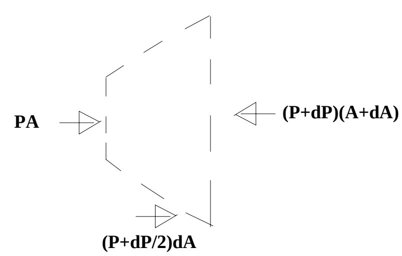
Simplifying and dividing through by ρAV

$$\frac{dP}{\rho}+\frac{dA}{A}+\frac{dV}{V}=0 \text{...........}(2.3)$$

Momentum equation for steady flow is

$$\sum \vec{F}=\int_C \int_S \vec{V}\left(\rho\vec{V}\bullet d\vec{A}\right) \text{...........}(2.4)$$

Only pressure forces act on the control volume



Equation (2.4) yields

$$\begin{aligned} PA &= \left(P+{dP}/{2}\right)dA-\left(P+dP\right)\left(A+dA\right) \\ &= \left(V+dV\right)\left(\rho+d\rho\right)\left(A+dA\right)\left(V+dV\right)-\rho V^2 A \text{.............}(2.5) \end{aligned}$$

Using equation (2.2) on the RHS of (2.5) and simplifying both sides:

$$dP + \rho V dV = 0 \ldots\ldots\ldots\ldots(2.6)$$

Energy equation:

$$\int_C \int_S \left(h + V^2/2\right)\rho V \bullet d\vec{A} = 0 \ldots\ldots\ldots\ldots(2.7)$$

$$\left[(h + dh) + \left(\frac{V + dV}{2}\right)^2\right]\left[(\rho + d\rho)(V + dV)(A + dA) - \left(h + V^2/2\right)\rho A V\right] = 0$$

Simplifying this equation we get

$$dh + dV^2/2 = 0 \ldots\ldots\ldots\ldots(2.8)$$

From the 1st and 2nd laws of thermodynamics,

$$Tds = dh - dP/\rho \ldots\ldots\ldots\ldots(2.9)$$

For isentropic flow: $dh = dP/\rho \ldots\ldots\ldots\ldots(2.10)$

Combining (2.10) with (2.8): $dP/\rho = -dV^2/2$

$$Or \;\; dP + \rho V dV = 0 \;\ldots\ldots\ldots\ldots\ldots(2.11)$$

Same as (2.6)

Combining (2.3) and 2.6) we obtain

$$dP + \rho V^2\left[\left(\frac{-d\rho}{\rho} - \frac{dA}{A}\right)\right] = 0................(2.12)$$

$$But\left(\frac{\partial P}{\partial \rho}\right)_S = C^2 \;\; i.e \;\; d\rho = \frac{dP}{C^2}............(2.13)$$

Substituting (2.13) in (2.12)

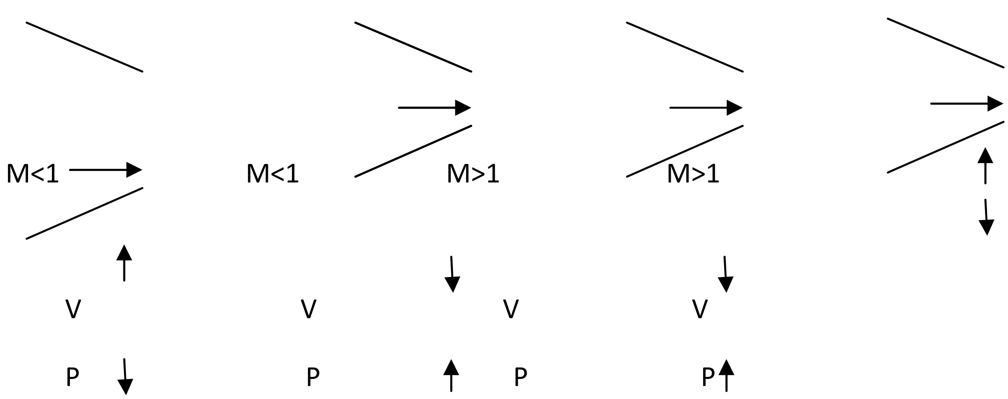
$$dP + \rho V^2\left(-\frac{dP}{\rho C^2} - \frac{dA}{A}\right)$$

$$But \;\; M = \frac{V}{C}$$

$$\Rightarrow (1 - M^2)dP = \rho V^2 \frac{dA}{A}.............(2.14)$$

Equation (2.14) demonstrates the influence of M on the flow. For M<1, $(1-M^2) > 0$, it implies that A increases given pressure increase using (2.6),

A increases → pressure increases → velocity decreases



Subsonic flow

Subsonic flow cannot be accelerated to velocity greater than the velocity of sound in a converging nozzle. Converging nozzle behaves like a diffuser in supersonic flow. This is true irrespective of the pressure difference imposed on the flow through the nozzle. If it is desire to accelerate a flow from negligible velocity to supersonic velocity, a convergent – divergent channel must be used.

Stagnation properties and the use of tables

Stagnation enthalpy (total enthalpy), h_t at a point in a flow is the enthalpy attained by bringing the flow adiabatically to rest at that point.

Equation 2.8 states $dh + dV^2/2 = 0$

Integrating both sides;

$$h + V^2/2 = Constant = h_t \text{ ....(2.15)}$$

Stagnation temperature (total temperature) is the temperature measured when the flow is brought adiabatically to rest at a point. For a perfect gas, $h_t = C_P T_t$

$$(h_t - h) = C_P (T_t - T)$$

Substituting in equation (2.15)

$$T_t = \frac{V^2}{2C_P} + T$$

$$\left(\frac{V^2}{2C_P T} + 1\right) T$$

$$But\ C_P = \frac{rR}{r-1}$$

$$T_t = \left[\left(1 + \frac{(r-1)}{2rRT} V^2\right)\right] T$$

$$T_t = T\left[1 + \frac{(r-1)}{2} M^2\right] \text{..........(2.16)}$$

Equation (2.16) is tabulated for r = 1.4 as $\left(\frac{T}{T_t}\right) vs\, M$

Example: If a perfect gas with r = 1.4 traveling at mach 3 with static temperature of 500k, the stagnation temperature is

$$T_t = T\left(T_t / T \right) = 500 \times 1 / 0.3571 = 1400K$$

Stagnation Pressure P_t at a point in a flow is defined as the pressure attained if the flow at that point is brought to rest isentropically.

For a perfect gas

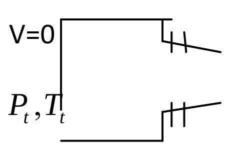
$$\frac{P_2}{P_1} = \frac{(T_2)^{r/r-1}}{T_1}$$

$$\frac{P_t}{P} = \frac{(T_2)^{r/r-1}}{T_1}$$

Using (2.16), *we obtain*

$$\frac{P_t}{P} = \left(1 + \frac{r-1}{2} M^2 \right)^{r/r-1} \quad \text{.........}(2.17)$$

$\frac{P}{P_t}$ *vs M is tabulated for* $r = 1.4$



As flow accelerates in the nozzle, static temperature and pressure decreases.

If flow is adiabatic,

$$h = V^2/2 = C_P T_t = Const.$$

I.e T_t remains constant (irreversibility will affect pressure and not temperature)

If flow is reversible as well as adiabatic, both P_t and T_t are constant in the nozzle and remain equal to the reservoir values

Static P and T

Mass flow rate

$$\dot{m} = \rho A V$$

$$\frac{P}{RT} AM (rRT)^{1/2}$$

$$Where\, P = \frac{P_t}{\left[\left(1 + \frac{(r-1)}{2} M^2\right)\right]^{r/r-1}}$$

$\dot{m}$ = ρAV

$= \frac{P}{RT} AM\, (rRT)^{1/2}$

$$and\;\; T = \frac{T_t}{\left(1 + \frac{r-1}{2} M^2\right)}$$

$$So\ that\, \dot{m} = \frac{P_t}{(RT_t)^{1/2}} A(r)^{1/2} \bullet M\left[\left(1 + \frac{r-1}{2} M^2\right)\right]^{r+1/2(1-r)} \;\;\ldots\ldots\ldots\ldots\ldots(2.18a)$$

$$Or\;\; \dot{m} = \frac{P_t A}{(RT_t)^{1/2}}\, f(r, M) \ldots\ldots\ldots\ldots.(2.18b)$$

$$Where\;\; f(r, M) = M(r)^{1/2}\left(1 + \frac{r-1}{2} M^2\right)^{r+1/2(1-r)}$$

For Isentropic flow, P_t and T_t are constant and cross sectional area of flow A can be related to M.

Select the area at which M=1, as a reference area A*.

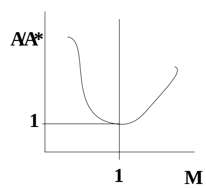
For steady flow,

$$\dot{m} A = \dot{m} A^*$$

$$So\ \ that\ \ \frac{P_t}{(RT)^{1/2}} Af(r,M) = \frac{P_t}{(RT)^{1/2}} A^* f(r)$$

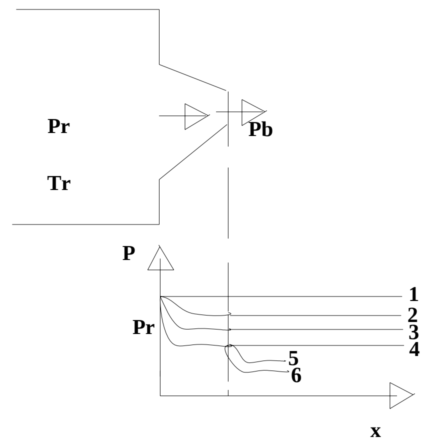
$$Where\ g(r,M) = \frac{(r)^{1/2} \frac{(r+1)^{r+1/2(1-r)}}{2}}{M(r)^{1/2}\left(1+\frac{r-1}{2}M^2\right)^{r+1/2(1-r)}} \ldots\ldots\ldots\ldots\ldots..(2.19)$$

Numerical values of A/A* vs M are shown in the table



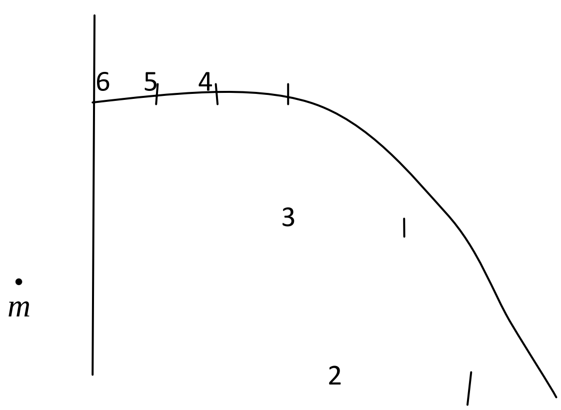
For each value of A/A* there are two possible isentropic solutions; one subsonic, and the other Supersonic. The minimum area or throat area occur at M = 1

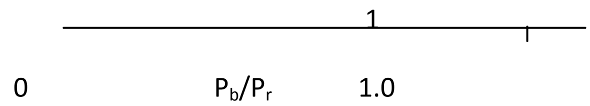
Isentropic flow in a converging nozzle;



Pressure distribution obtained in the nozzle for 6 different values of P_b

Fluid stored in a large reservoir is to be discharged through a converging nozzle. For a constant reservoir pressure Pr, determine ṁ through nozzle as a function of the back pressure, P_b of imposed on the nozzle.





Mass flow rate through the nozzle vs P_b/P_r

P_b = Stagnation pressure at the ambient air around nozzle exit.

For $P_b = P_r$ (curve 1), there is no flow in the nozzle as P is in-variant with x. As P_b is reduced below P_r, $\dot{m}$ increases and P reduces with x for the subsonic flow. The velocity at the nozzle exit plane increases as P_b is reduced until the velocity of sound is reached at the exit plane curve 4. With further reduction in P_b, $\dot{m}$ remains constant. With P_b/P_r greater than or equal to that corresponding to (curve 4), flow is able to sense reduction in back pressure and adjust so that static pressure at the exit plane equal the back pressure. With P_b/P_r less than that corresponding to curve 4, flow is unable to sense the reduction in back pressure, so the flow through the nozzle remains as it was in curve 4. Therefore the exit plane pressure is greater than P_b and the flow must adjust to the back pressure by means of an expansion occurring outside the nozzle. Reduction in P_b below that of curve 4 cannot cause any more flow to be induced through the nozzle. Under these conditions, the nozzle is said to be choked.

$$P_t = P\left(1+\frac{r-1}{2}M^2\right)^{r/r-1}$$

To just choked the nozzle, M=1 at exit plane

$$\frac{P_r}{P_b} = \left(1+\frac{r-1}{2}\right)^{r/r-1}$$

For r = 1.4, P_b/P_r = 0.5383

This ratio below which the nozzle is choked is termed the critical pressure ratio.

<u>Reference speeds</u>

Maximum velocity corresponding to a given stagnation temperature is obtained when the absolute temperature is zero.

$$h_t = C_P \overset{\to 0}{T} + \frac{1}{2}V^2$$

$$V_{\max} = \left[\frac{2r}{r-1} RT_t\right]^{\frac{1}{2}} ....(2.20)$$

Since T = 0, absolute, V = V_{max}, corresponding M →∞ since C = 0

Another useful reference velocity is the speed of sound at the stagnation temperature.

$$C_t = (rRT_t)^{\frac{1}{2}} ....(2.21)$$

A 3rd convenient reference velocity is the critical speed i.e velocity at Mach number unity.

$$V^* = C^* ....(2.22)$$

From steady flow energy equation

$$V = \left[2C_P(T_t - T)\right]^{\frac{1}{2}} = \left[\frac{2r}{r-1} R(T_t - T)\right]^{\frac{1}{2}}$$

$$V^* = \left[2C_P(T_t - T^*)\right]^{\frac{1}{2}} = \left[\frac{2r}{r-1} R(T_t - T^*)\right]^{\frac{1}{2}} ....(2.23)$$

$$V = \left[\frac{2r}{r-1} R(T_t - T^*)\right]^{\frac{1}{2}} = (rRT^*)^{\frac{1}{2}} ....(2.24)$$

Which gives $T^*/T_t = 2/r + 1$ -------------- (2.25)

Substituting for T* in (2.23)

$$V^* = C^* = \left[\frac{2r}{r+} RT_t\right]^{\frac{1}{2}} ....(2.26)$$

We can get the following relation between the three reference velocity

$$\frac{C^*}{C_t} = \left(2/_{r+1}\right)^{1/2} = 0.913 \ for \ r = 1.4$$

$$\frac{V_{max}}{C_t} = \left(2/_{r+1}\right)^{1/2} = 2.24 \ for \ r = 1.4$$

$$\frac{V_{max}}{C^*} = \left[\left(\frac{r+1}{r-1}\right)\right]^{1/2} = 2.45 \ for \ r = 1.4$$

$$2h_t = V^2 + \frac{2}{r-1}C^2 = \frac{2C_t^2}{r-1} = V_{max}^2 = \frac{r+1}{r-1}C^{*2} \ ....(2.27)$$

The dimensionless velocity M*

$$M^* = \frac{V}{C^*}$$

M* has 2 disadvantages;

1. It is not proportional to the velocity alone

2. At high speed, it tends towards infinity; therefore it is often useful to work with a dimensionless quantity obtained through dividing the flow velocity V by one of the 3 reference velocities. The most useful of this is:

M* = V/C* = V/V*

N.B

M* is not the value of M at the local sonic condition like V*, P*, T* etc but is rather defined as given above.

From definition of M and M*

$$M*^2=\frac{V^2}{C*^2}=\frac{V^2}{C^2}\bullet\frac{C^2}{C*^2}=\frac{M^2C^2}{C*^2}\text{...............}(2.28)$$

From (2.27)

$$\frac{V^2}{C*^2}+\frac{2}{r-1}\frac{C^2}{C*^2}=\frac{r+1}{r-1}\text{.................}(2.29)$$

Eliminating $C^2/C*^2$ from this pair of equations and rearranging

$$M*^2=\frac{{}^{r+1}\!/_{2}M^2}{1+{}^{r-1}\!/_{2}M*^2}\text{............}(2.30)$$

Or

$$M*^2=\frac{{}^{2}\!/_{r+1}M*^2}{1-{}^{r-1}\!/_{r+1}M*^2}\text{............}(2.31)$$

When M < 1, then M* < 1

When M = 1, then M* = 1

When M > 1, then M* > 1

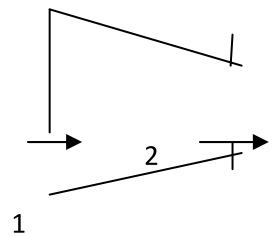
When M = 0, then M* = O

When M →∞, then $M*\rightarrow\left[\frac{(r+1)}{(r-1)}\right]^{1/2}$

<u>Worked Example</u>

An air stream flowing in a converging duct shown below; From a cross sectional area A_1 of 1.5m^2 to a cross sectional area A_2 of 0.10 m^2. If T_1=500k, P_1=150Kpa, V_1=100m/s find M_2, P_2 and T_2, assume steady 1-D isentropic flow.

Solution



$$M_1 = \frac{V_1}{C_1}$$

$$C = (rRT)^{1/2}$$

$$M_1 = \frac{100}{(1.4 \times 287 \times 500)^{1/2}}$$

$$= 0.223$$

$$From\ table\ B_2\ at\ M = 0.223,$$

$$\frac{A_1}{A^*} = 2.7076,$$

$$Thus\ A_1 = 2.7076A^*$$

$$But\ \frac{A_2}{A_1} = \frac{10}{15}$$

$$\therefore \frac{A_2}{A_1} = \frac{10}{15} \times 2.7076 = 1.8051$$

$$From\ table\ B_2, M_2 = 0.34$$

For isentropic flow, P_t and T_t are constant.

At M = 0.223, $P_1/P+1$ = 0.96685, T_1/T_{t1} = 0.99041

P_{t1} =150/0.99041 = 504.841KPa

T_{t1} = 500/0.99041 = 504.841KPa

At M_2 = 0.34

P_2/P_{t2} = 0.92312

Thus P_2 = 0.92312 x 155.143

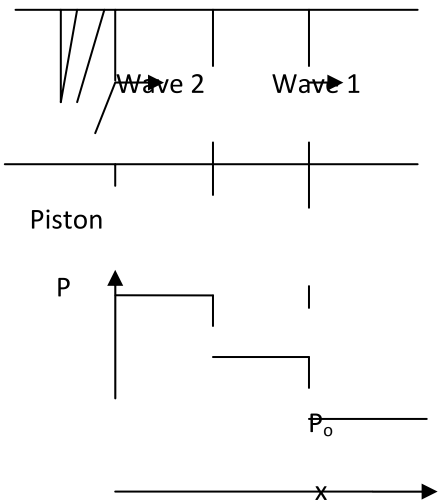
= 143.2156KPa

T_2/T_{t2} = 0.97740

Thus T_2 = 0.97740 x 50

= 493.4316K

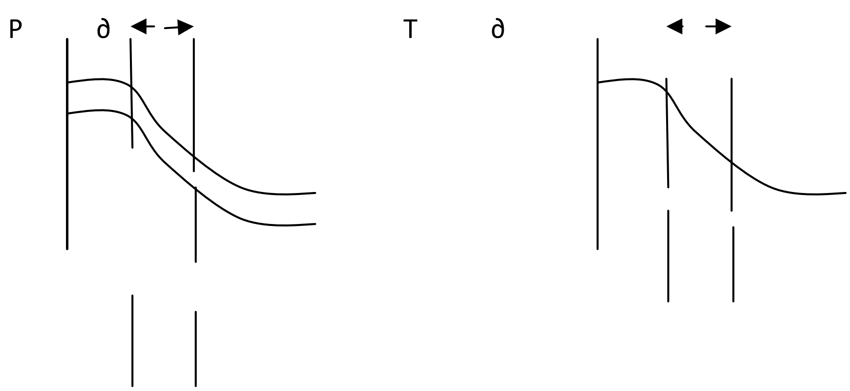
CHAPTER 3: NORMAL SHOCK WAVES (NSW)



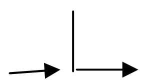
Piston given velocity ∂V to the right will give wave 1. Piston given increment in Velocity ∂V to the right gives wave 2. Wave 2 travels at higher absolute velocity than wave 1 because gas in which wave 2 travels in warmer than that in which wave 1 travels. $C = (rRT)^{1/2}$

Wave 2 signal travels at velocity C relative to gas particles moving to the right at velocity ∂V whereas wave 1 travel through gas moving at zero velocity.

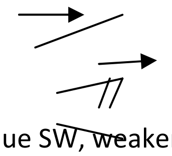
Suppose piston is given a finite velocity increment ΔV to the right. We can think that ΔV as being made up of a large number of infinitesimal increments ∂V → shock wave.



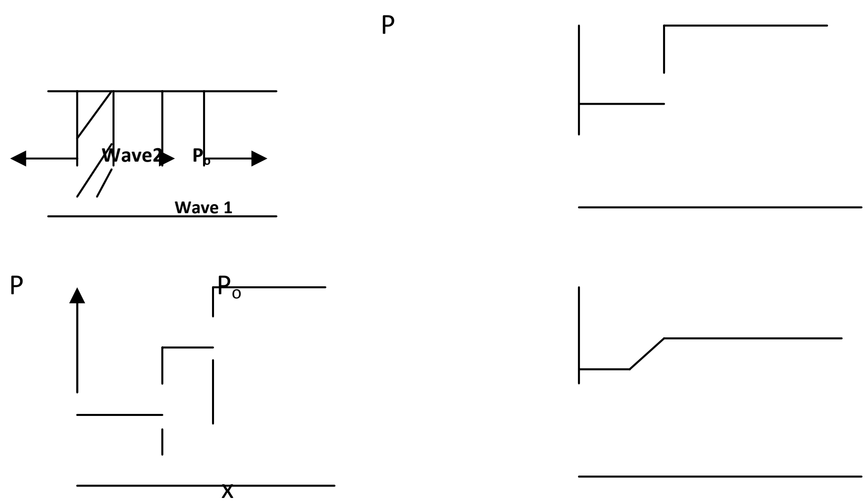
∂ = shock thickness ≈ 10^{-5}



NSW



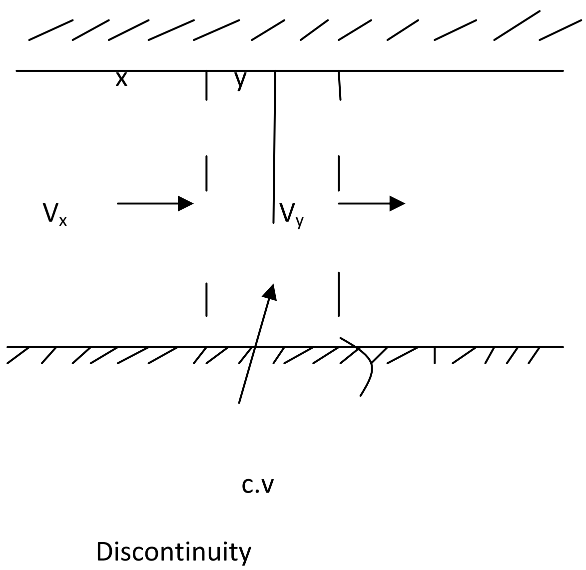
Oblique SW, weaker than NSW



For the case when piston is given velocity increment ΔV to left, we can think of it as very large number of dV to the left. The wave smears out, i.e becomes less steep. Thus the creation of a finite expression shock wave is impossible.

<u>Equation of motion</u>

The processes taking place in the shock waves are extremely complex. Temperature and velocity gradients internal to the shock provide heat conduction and viscous dissipation that render the shock process internal irreversible. We can choose a control volume in such a way that we can write the flow equation without regard to the complexity of the internal processes in the shock waves.



SFEE for the adiabatic process between X and Y

$$H_x + V_x^2/2 = h_y + V_y^2/2 = h_t \text{ --------- } (3.1)$$

Where h_t is the stagnation enthalpy on both sides of the NSW.Cross sectional area is the same on both sides of the NSW since it is thin.

Continuity equ

$$\dot{m}/A = \rho_x V_x = \rho_y V_y \text{ ------------------------------------- } (3.2)$$

Applying the momentum theorem to the flow through the NSW

$$P_x - P_y = \dot{m}/A \left(V_y - V_x\right) \ldots\ldots\ldots\ldots (3.3)$$

Combing (3.2) and (3.3)

$P_x + \rho_x V^2_x = P_y + \rho_y V^2_y$ ----------------- (3.4)

Equation of state for the fluid may be wirtten implicitly in the form

$h = h(s, \rho)$ ---------------- (3.5a)

$s = s(P, \rho)$ ---------------- (3.3b)

NSW in a perfect gas

For a perfect gas with constant specific heats undergoing an adiabatic process:

$C_pT_x + V^2_x/2 = C_pT_y + V^2_y/2 = C_pT_t$ ---------------- (3.6)

Or $T_{tx} = T_{ty}$

Equation (3.6) simplifies to

$$T_x\left(1+\frac{r-1}{2}M_x^2\right) = T_y\left(1+\frac{r-1}{2}M_y^2\right)\text{.................}(3.7)$$

Equation (3.4) becomes

$$P_x + \rho_x V^2 = P_x\left(1+\rho_x\frac{V_x^2}{P_x}\right)$$
$$= P_x\left(1+rM_x^2\right)$$
$$Or$$
$$P_x\left(1+rM_x^2\right) = P_y\left(1+rM_y^2\right)\text{.................}(3.8)$$

Combining (3.8), (3.7) and (3.2)

$\rho_x V_x = \rho_y V_y$

$$\frac{P_x}{RT_x} M_x (rRT_x)^{1/2} = \frac{P_y}{RT_y} M_y (rRT_y)^{1/2}$$

We can simplify this to get

$$\frac{M_x}{1+rM_x^2}\left[\left(1+\frac{r-1}{2}\right)M_x^2\right]^{1/2}$$

$$= \frac{M_y}{1+rM_y^2}\left[\left(1+\frac{r-1}{2}\right)M_y^2\right]^{1/2} \text{ .... (3.9)}$$

By inspection, a trivial solution to equation (3.9) is M_y = M_x. This solution corresponds to isentropic flow in a constant area duct. From (3.8) and (3.7)

$P_y = P_x$ and $T_y = T_x$

Equation (3.9) can be solved to yield M_y in terms of M_x.

Squaring both sides of (3.9)

$$\frac{M_x^2\left(1+\frac{r-1}{2}M_x^2\right)}{\left(1+rM_x^2\right)^2} = \frac{M_y^2\left(1+\frac{r-1}{2}M_y^2\right)}{\left(1+rM_y^2\right)^2} \text{............} (3.10)$$

Express in terms of quadratic in M^2_y

$$M_y^4\left[\frac{(r-1)}{2} - r^2 L\right] + M_y^2(1-2rL) - L = 0$$

$$Where\ L = \frac{M_x^2\left(1+\frac{r-1}{2}M_x^2\right)}{\left(1+rM_x^2\right)^2}$$

Solving the quadratic equation for M_y

$$M_y^2 = \frac{M_x^2 + {}^{2}\!/\!_{(r-1)}}{{}^{2r}\!/\!_{(r-1)} M_x^2 - 1} \quad \ldots\ldots\ldots\ldots\ldots(3.11)$$

For $M_x > 1$, M_y is less than one and vice versa.

From equation (3.8)

$$\frac{P_y}{P_x} = \frac{\left(1 + rM_x^2\right)}{\left(1 + rM_y^2\right)}$$

I.e $M_x > 1$ corresponds to the case of compression shock, and $M_x < 1$ is the case of expansion shock.

Therefore the flow at x is supersonic while the flow at y is subsonic.

Substituting for M_y from (3.11) in (3.7)

$$\frac{T_y}{T_x} = \frac{\left(1 + \frac{r-1}{2} M_x^2\right)\left(\frac{2r}{r-1} M_x^2 - 1\right)}{\frac{(r+1)^2 M_x^2}{2(r-1)}} \quad \ldots\ldots\ldots\ldots\ldots(3.13)$$

Substituting for M_y from (3.11) in (3.8)

$$\frac{P_y}{P_x} = \frac{2r}{r+1} M_x^2 - \frac{(r-1)}{r+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.13)$$

The density ratio may be found from (3.12) and (3.13) and the perfect gas below:

$$\frac{P_y}{P_x}=\frac{\left(P_y/P_x\right)}{\left(T_y/T_x\right)}=\frac{(r+1)M_x^2}{(r-1)M_x^2+2}\ldots\ldots\ldots\ldots..(3.14)$$

The ratio of stagnation pressure is a pressure of the irreversibility in the shock process. It may be found by observing that

$$\frac{P_{ty}}{P_{tx}}=\left(\frac{P_{ty}}{P_y}\right)\bullet\left(\frac{P_y}{P_x}\right)\bullet\left(\frac{P_x}{P_{tx}}\right)\ldots\ldots\ldots.(3.16)$$

$$But\,\frac{P_t}{P}=\left(1+\frac{r-1}{2}M^2\right)^{r/r-1}\ldots\ldots\ldots..(3.17)$$

Substituting for P_{ty}/P_y and P_x/P_{tx} from (3.17) and P_y/P_x from (3.13) in (3.16), we get after algebraic simplification

$$\frac{P_{ty}}{P_{tx}}=\frac{\left[\dfrac{(r+1)M_x^2/2}{1+r-1/2\,M_x^2}\right]^{r/r-1}}{\left[\left(2rM_x^2\right)/r+1-(r-1)/(r+1)\right]^{1/r-1}}\ \ldots\ldots\ldots..(3.18)$$

Entropy change across the shock wave

$$S_y - S_x = C_P \ln\left(T_y / T_x\right) - R \ln\left(P_y / P_x\right)$$

$$S_y - S_x = C_P \ln\left[\frac{T_y / T_x}{\left(P_y / P_x\right)^{r-1/r}}\right] ....(3.19)$$

And also since $\frac{T_t}{T} = \left[1 + \frac{r-1}{2} M^2\right]$ *and* $\frac{P_t}{P} = \left[1 + \frac{r-1}{2} M^2\right]^{r/r-1}$

The entropy change can be written as

$$S_y - S_x = C_P \ln\left[\frac{\left(T_{ty} / T_{tx}\right)}{\left(P_{ty} / P_{tx}\right)^{r-1/r}}\right] ....(3.20)$$

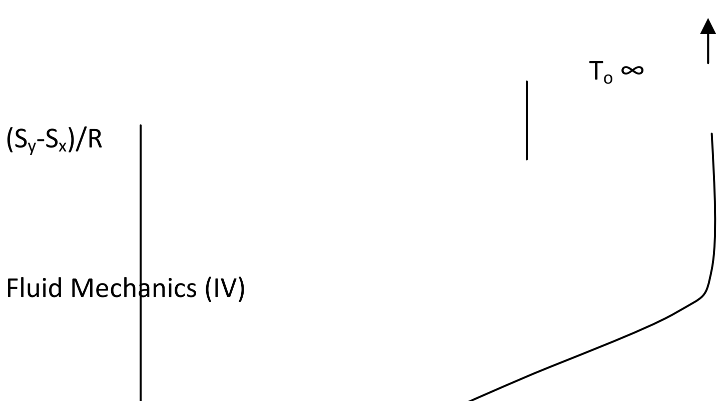
$$Since\, T_{ty} = T_{tx}$$

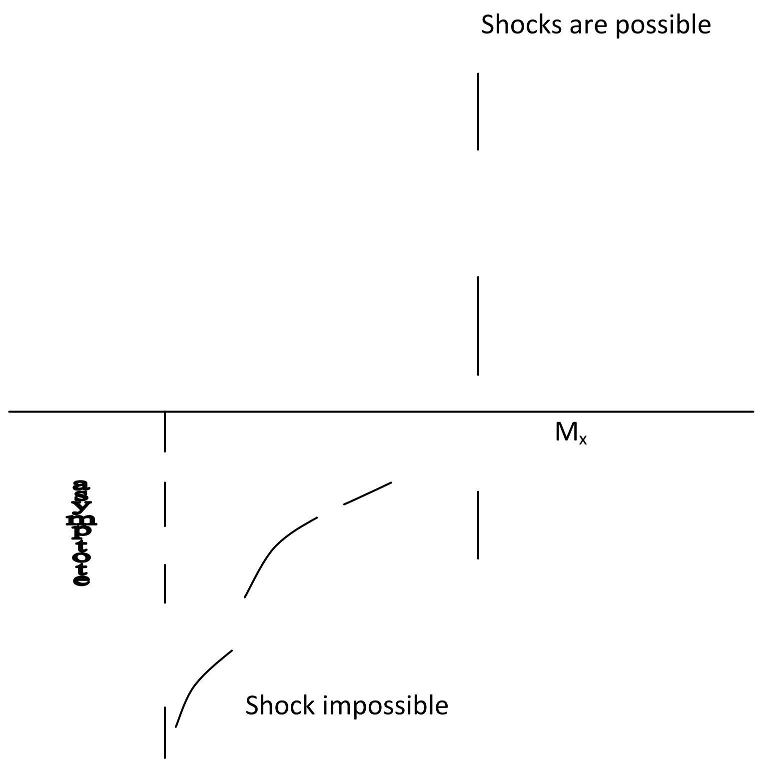
$$\frac{\left(S_y - S_x\right)}{R} = -\ln\left(\frac{P_{ty}}{P_{tx}}\right) ....(3.21)$$

After substituting for P_{ty}/P_{tx} from (3.18) in (3.21), we will obtain;

$$\frac{\left(S_y - S_x\right)}{R} = \frac{r}{r-1} \ln\left[\left(\frac{2}{r+1}\right) M_x^2 + \frac{r-1}{r+1}\right] + \frac{1}{r-1} \ln\left[\left(\frac{2r}{r+1}\right) M_x^2 - \frac{r-1}{r+1}\right] ....(3.22)$$

A careful study of (3.22 indicates for gas with $1 < r < 1.67$, the entropy change is always positive when $M_x > 1$ and is always negative when $M_x < 1$





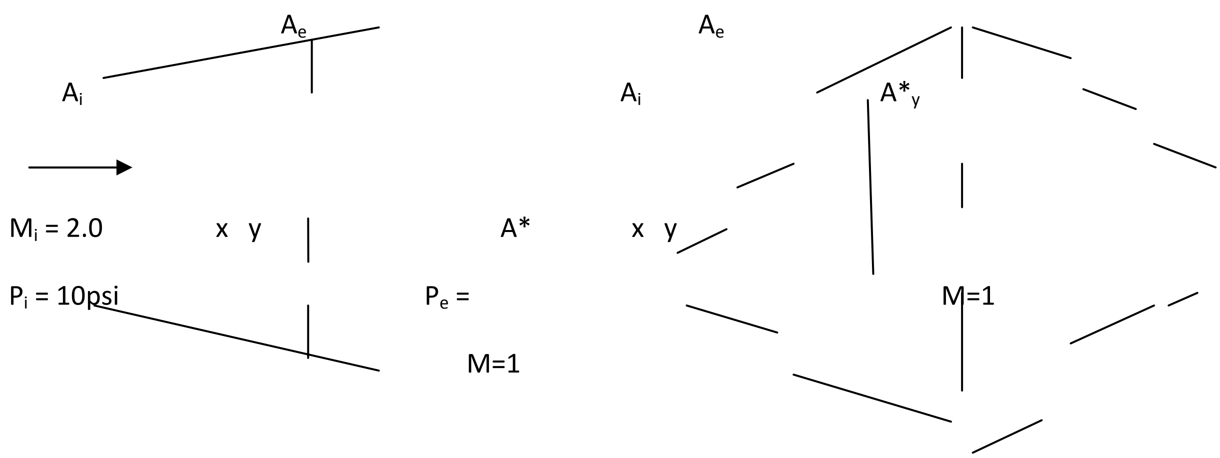
$M_x < 1$ gives shock rarefaction (or expansion shock). It is thus shown that an expansion shock is impossible, since the 2nd law of thermodynamics for this $(S_y - S_x) < 0)$ for adiabatic process.

Note: Equ (3.12), (3.13), (3.14) and (3.18) are presented in table B3

1psi = 6.7KPa

Example

An upstream at mach 2 and pressure 10psi absolute and temperature 400K enters a diverging channel with a ratio of exit area to inlet area of 3.0. Determine the back pressure necessary to produce a normal shock in the channel at an area equal to twice the inlet area. Assume 1-D steady flow with the air behaving as a perfect gas with constant specific heat. Assume isentropic flow, except for the normal shock.



$$\dot{m}_{A^*x} = \dot{m}_{A^*y}$$

$$\frac{P_{tx} A^*_x}{(RT_{tx})^{1/2}} \bullet f(r, M) = \frac{P_{ty} A^*_y}{(RT_{ty})^{1/2}} \bullet f(r, M)$$

But M = 1 at A^*_x and A^*_y

$$\Rightarrow T_{tx} = T_{ty}$$

$$\therefore P_{tx} A^*_x = P_{ty} A^*_y$$

From table B2, At M = 2.0

$A_i/A^*_x = 1.69$

$A_x/A^*_x = (A_x/A_i).\ (A_i/A^*_x) = (2.0)\ (1.69) = 3.39$

$M_x = 2.76$

Using table B3 for NSW

At $M_x = 2.76$, $P_{ty}/P_{tx} = 0.403$

$A^*_x/A^*_y = 0.403$

$A_e/A^*_y = (A_x/A_i). (Ai/A^*_x). (A^*_x/A_y)$

$=3.0 \times 1.69 \times 0.403$

$= 2.04$

Using table B2

$M_e = 0.3$

From B3, $M_y = 0.491$

Exit pressure

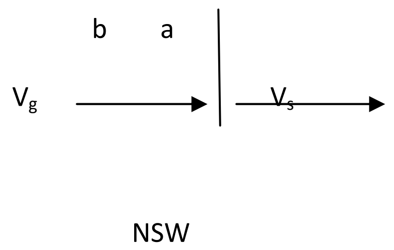
$P_e/P_i = (P_e/P_{ty}). (P_{ty}/P_{tx}).(P_{tx}/P_i)$

$= (0.940) (0.403) (1/0.128)$

$P_e = 29.61$ psia

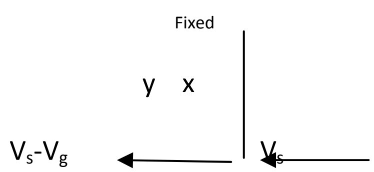
Moving Normal shock

Many physical situations arise in which normal shock wave is moving, e.g. when an explosion occurs, re-entry of a blunt body from space, closing a gas line valve suddenly.



Normal shock wave propagating in still air at V_s

V_g = velocity of gases behind the wave



Observer "sitting" on the SW

Consider the effects of observer velocity on static and stagnation properties. Static properties are measured with an instrument moving at the absolute flow velocity. Thus they are independent of the observer's velocity.

$P_y/P_x = P_b/P_g$, $T_y/T_x = T_b/T_a$

Stagnation properties are measured by bringing the flow to rest.

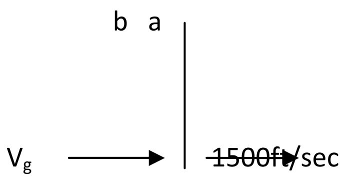
$T_x = T_a$ $P_x = P_a$

$T_{tx} > T_{ta}$, and $P_{tx} > P_{ta}$

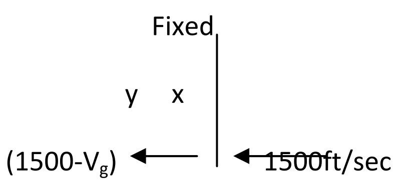
Example

A normal shock moves at a constant velocity of 1500 ft/s into still air at 0^0F and 10 psia. Determine the static and stagnation condition present in the air atter passage of the wave.

Solution



For a stationary observer



For observer riding on the wave

$M_x = 1500/(rRTg_c)^{1/2}$

$T^0R = {}^0F + 459.67$

T in ranking = 459.67, $g_c = 32.2 ft/sec^2$, $R = 53.3 Btu/lb^0R$

$M_x = 1.43$

$T_y/T_x = 1.27$, $T_2 = 584^0R$

$P_y/P_x = 2.22$, $P_2 = 22.2psia$

$P_y/P_x = 1.74$

From $P_xV_xA_x = P_yV_yA_y$

$V_y/V_x = (1500 - V_g)/1500 = 1/1.74$

$1500 – V_g = 861ft/sec$

$V_g = 639ft/sec$

Since the velocity of the observer does not affect static properties

$P_b = 22.2psia = P_y$

$T_b = 584^0R = T_y$

$$M_b = \frac{V_g}{(rRg_cT_b)^{1/2}} = 0.54$$

At $M_b = 0.54$

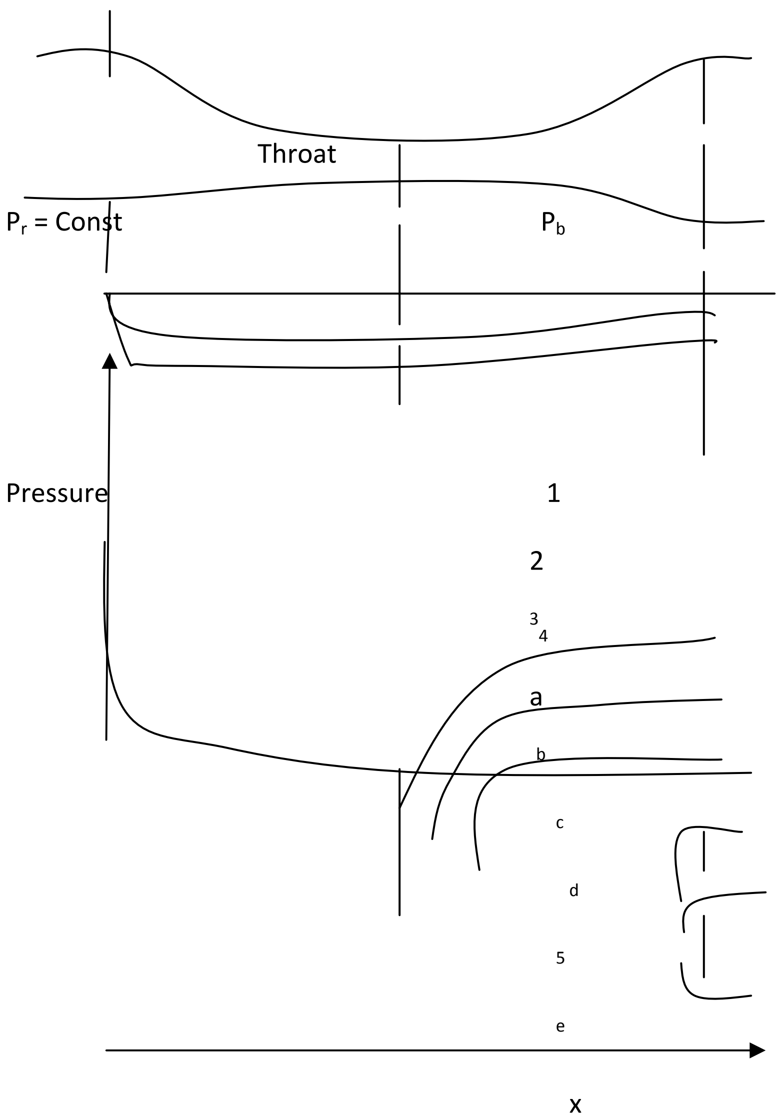
$T/T_t = 0.945$, $P/P_t = 0.82$

$P_{tb} = 22.2/0.82 = 27.1psia$

$T_{tb} = 584/0.945 = 619^0R$

Application

1. **Performance of converging – Diverging Nozzles**



The nozzle may be attached at the inlet to a high pressure reservoir and allowed to discharge into the atmosphere; alternatively, it may be attached at the outlet to a vacuum tank and allowed to draw its supply from the atmosphere.

P_r is fixed

Find the pressure distribution in the nozzle for various values of back pressure P_b

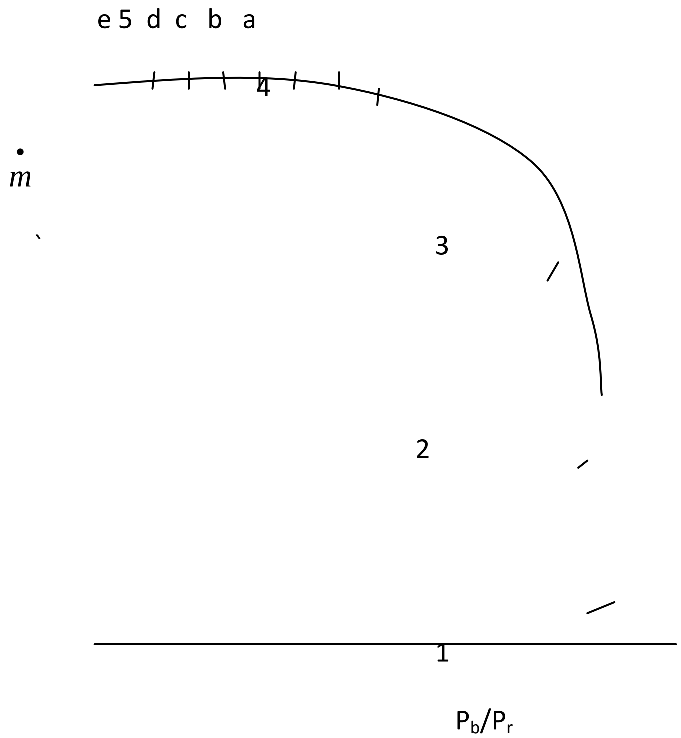
With $P_b = P_r$ (curve 1) there is no flow in the nozzle.

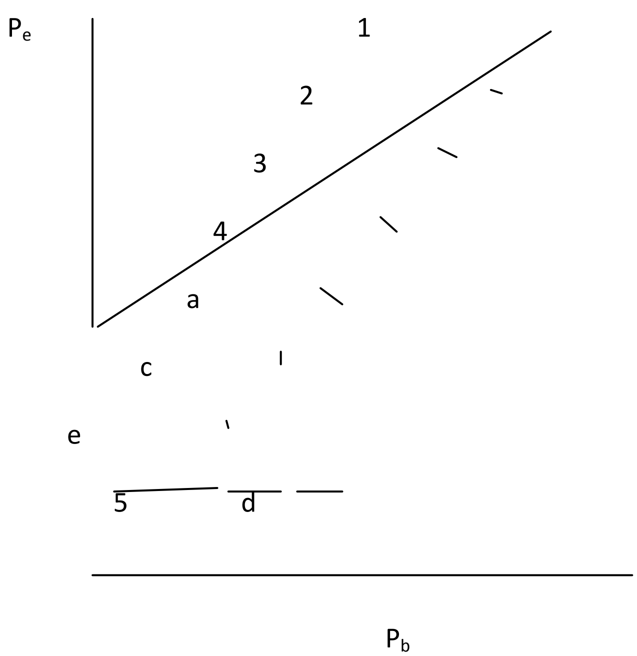
When P_b is a little less than P_r, subsonic flow is induced through the nozzle with pressure decreasing to the divergent portion of the nozzle (curve 2.3). When the back pressure is lowered to that of curve 4, sonic flow occurs at the nozzle throat. Further reduction in back pressures can no more increase the mass flow rate through the nozzle. As the back pressure is reduced below that of curve 4 , a normal shock appear in the nozzle just downstream of the throat(curve a).Further reduction in back pressure causes the shock to move downstream (curve b) until for a low enough back pressure the normal shock position itself at the nozzle exit plane (curve c). As the back pressure is lower below that of curve c, an oblique shock wave appears at the exit plane.

Further reduction in P_b causes the oblique wave to bend further away from the flow direction, thus decreasing the shock strength until eventually the isentropic case is reached (curve 5).

Curve 5 corresponds to the design condition in which the flow is perfectly expanded in the nozzle to the back pressure.

For back pressure below that of curve 5, exit plane pressure is greater than the back pressure. A pressure decrease occurs outside the nozzle in the form of an expansion wave.


e 5 d c b a
4
$\dot{m}$
3
2
1
P_b/P_r


P_e
1
2
3
4
a
c
e
5
d
P_b

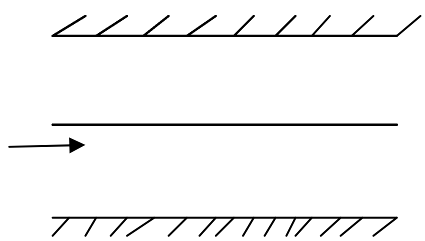
Nozzle is said to be over expanded when $P_e < P_b$, e.g between c and 5. Nozzle is said to be under expanded when $P_e > P_b$, e.g pressure below curve 5.

CHAPTER 4:FLOW IN CONSTANT AREA DUCT WITH FRICTION (FANNO LINE FLOW)

Assumptions:

- 1-D flow
- There is friction at the wall only (no internal friction)
- Wall friction is chief factor bringing about changes in flow properties.
- No heat transfer to or from the stream (approx. true for short ducts).
- No area charge along the duct

Fanno line flow = Adiabatic flow with as external work.



Q = O, A = constant.

Energy equation, $h + V_2/2$ = Const = h_t ---------------- (4.1)

Continuity equation, ρV = Const -------------------- (4.2)

From 1st and 2nd laws of thermodynamic

$Tds = dh - dP/\rho = du - (P/\rho^2)\, dP$ ---------------------- (4.3)

For a perfect gas

$P/\rho = RT$

Therefore ; $ds = du/T - Rdp/p$ --------------- (4.4)

Assuming constant specific heats with state i as reference state in the flow.

$$S - S_1 = C_V \ln\left(\frac{T}{T_1}\right) - R\ln\frac{\rho}{\rho_1} \ldots\ldots\ldots\ldots (4.5)$$

Using (4.2)

$$S - S_1 + C_V \ln\left(\frac{T}{T_1}\right) = R \ln \frac{V}{V_1} \ldots\ldots\ldots\ldots(4.6)$$

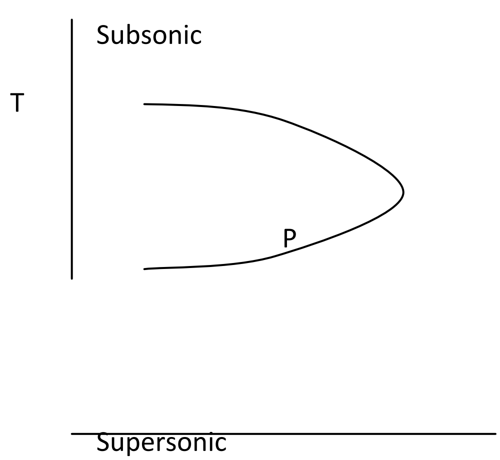
From (4.1)

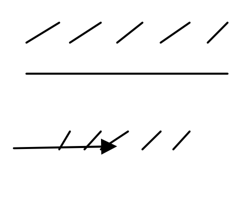
$$\vec{V} = 2(h_t - h)^{1/2}$$
$$= \left[2C_P(T_t - T)\right]^{1/2} \ldots\ldots\ldots\ldots\ldots(4.7)$$

Substitute for $\vec{V}$ from (4.7) in (4.6) and also for C_p/R in terms of r

$$\frac{(S - S_1)}{C_V} = \ln\left(\frac{T}{T_1}\right) + \frac{r-1}{2}\ln\left[\frac{(T_t - T)}{(T_t - T_1)}\right]$$
$$= \ln T + \frac{r-1}{2}\ln(T_t - T) + Const\ldots\ldots\ldots\ldots(4.8)$$

Equation (4.8) can be plotted on a T-S diagram. The line obtained is called a Fanno line.





S

Consider point P at P, ds/dt = 0

Differentiating equation (4.8)

$$\frac{d}{dT}\left(\frac{S}{C_v}\right)=\frac{1}{T}-\frac{(r-1)}{2(T_t-T)}=0\ at\ P=0$$

$$\therefore \frac{1}{T}=\frac{(r-1)}{2(T_t-T)}$$

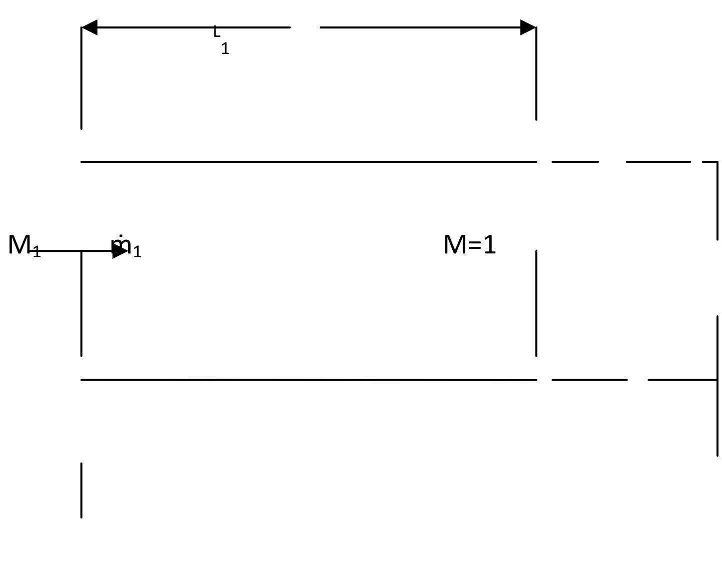
$$2(T_t-T)=(r-1)T$$

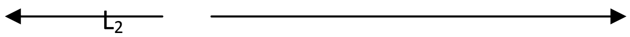
$$But\, C_P(T_t-T)=\frac{V^2}{2}$$

$$So\ that\ \frac{V^2}{C_P}=(r-1)T$$

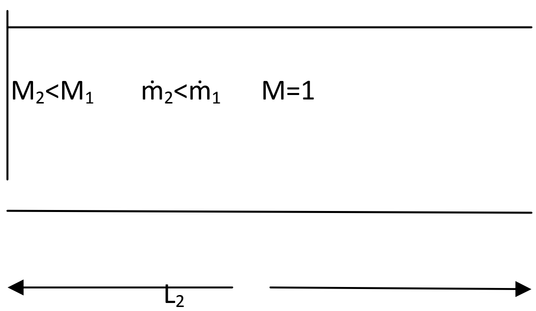
$$Or\, V^2=C_P(r-1)T=rRT\ \ at\ \ P, M=1$$

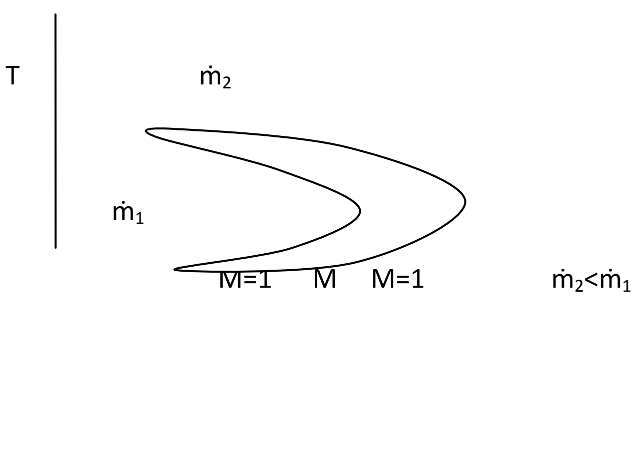
Thus in the T-S diagram the state of the fluid continually moves to the right till maximum entropy is attained at the state corresponding to P. For subsonic flow mach number increase with axial distance to 1. For supersonic flow, the entrophy must again increase so that M decreases to 1 at P. Suppose the duct is long enough for a flow initially subsonic to reach mach 1, and an additional length is added as shown below:





The flow Mach number for the given mass flow rate cannot go pass 1 without decreasing entropy. This is impossible from 2[nd] law of thermodynamic; hence an additional length brings about the reduction in mass flow rate. The flow jumps to another fanno line as shown below.

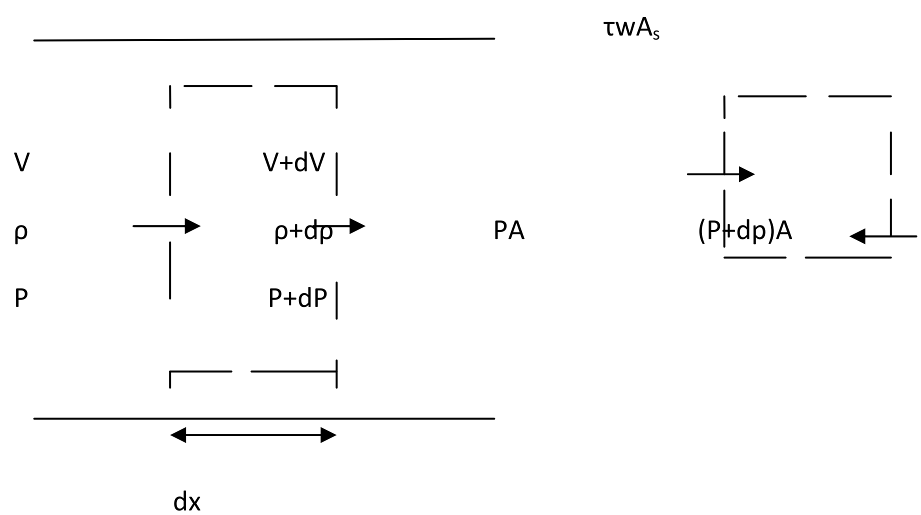




Corresponding to a given inlet subsonic M, there is a certain maximum duct length 1max, beyond which a flow reduction occur.

Suppose the inlet flow is supersonic and the duct length is made greater than lmax required to produce M = 1 at the exit, the flow adjust to the additional length by means of a normal shock rather than a flow reduction. The location of the shock in the duct is determined by the back pressure imposed on the duct.

Determination of change of properties with actual duct length



τ_W = shear stress due due to wall friction

A_s = lateral surface area over which the friction acts

A = cross sectional area of duct.

Momentum equation for steady flow

$$\sum F_x = \int_C \int_S V_x\left(\rho \vec{V} \bullet d\vec{A}\right)$$

$$PA - (P + dP)A - \tau_W A_S = \rho AV(V + dV) - (\rho AV) \bullet V \text{ ....}(4.9)$$

Define hydraulic diameter

$D_h =$ 4A/perimeter = four times the c.s.a of flow/wetted perimeter

For a circular duct

$$D_h = 4\frac{\pi D^2 / 4}{\pi D} = D$$

For a square duct of side S

$$D_h = 4S^2 / 4S = S$$

$$A_S = 4A / D_L \, dx = Perimeter \bullet dx \text{........}(4.10)$$

Substituting (4.10) in (4.9)

$$-AdP - \tau_W dx \, 4A / D_h = \rho AVdV \text{.........}(4.11)$$

Coeff of friction

$$f = \frac{\tau_W}{\frac{1}{2}\rho V^2}$$

$$f = f(\mathrm{Re}, wall\,roughness)$$

$$\tau_W = \frac{1}{2}\rho V^2 f \,..........(4.12)$$

Substituting for τ_w from (4.12) in (4.11)

$$-AdP - 4\rho\frac{V^2}{2} f(dx)\frac{A}{D_h} = \rho AVdV \,............(4.13)$$

Dividing both sides by ρA

$$\frac{dP}{P} + \frac{rM^2}{2} \bullet 4f\frac{dx}{D} + rM^2\frac{dV}{V} = 0 \,..............(4.14)$$

Since $D_h = D$ *for circular pipe,*

Continuity equation, $\rho V = Const$ *which also gives*

$$\frac{d\rho}{\rho} + \frac{dV}{V} = 0 \,................(4.15)$$

From perfect gas equation of state

$$\frac{dP}{P}=\frac{d\rho}{\rho}+\frac{dT}{T}.............(4.16)$$

$$Also\ M=\frac{V}{(rRT)^{1/2}}$$

Logarithmic differentiation of this equation

$$\frac{dM}{M}=\frac{dV}{V}-1/2\frac{dT}{T}...............(4.17)$$

From (4.16) *and* (4.15)

$$\frac{dP}{P}=\frac{-dV}{V}+\frac{dT}{T}$$

Substituting for $\frac{dV}{V}$ *from* (4.17)

$$\frac{dP}{P}+\frac{dM}{M}+1/2\frac{dT}{T}=\frac{dT}{T}$$

$$Or\ \frac{dP}{P}=-\frac{dM}{M}+1/2\frac{dT}{T}................(4.18)$$

Substituting for dP/P from (4.18) and dV/V from (4.17) in (4.14)

$$-\frac{dM}{M}+1/2\frac{dT}{T}+\frac{rM^2}{2}.4f\frac{dx}{D}+rM^2\left(\frac{dM}{M}+1/2\frac{dT}{T}\right)=0$$

$$-\frac{dM}{M}+1/2\frac{dT}{T}+\frac{rM^2}{2}.4f\frac{dx}{D}+rM^2\frac{dM}{M}+rM^2\frac{dT}{T}=0..........(4.19)$$

$$But\ T\left[\left(1+\frac{r-1}{2}M^2\right)\right]=Const.$$

$$\ln T+\ln\left[\left(1+\frac{r-1}{2}M^2\right)\right]=Const.$$

$$Or\ \frac{dT}{T}+\frac{(r-1)M^2\,dM/M}{1+(r-1)/2\,M^2}=0...............(4.20)$$

Substitute for dT *from* (4.20) *in* (4.19)

$$\left(1/2+rM^2/2\right)\left[-\frac{(r-1)M^2\,dM/M}{1+r-1/M^2}\right]-\frac{dM}{M}+rM^2\frac{dM}{M}+\frac{rM^2}{2}.4f\frac{dx}{D}=0$$

Combining terms

$$4f\frac{dx}{D}=\left[\frac{2\,dM/M\left(1-M^{2}\right)}{-\left(1+r-1/2\,M^{2}\right)rM^{2}}\right]\text{..............}(4.21a)$$

$$4f\frac{dx}{D}=\frac{1-M^{2}}{rM^{4}\left[1+r-1/2\,M^{2}\right]}dM^{2}\text{ ....}(4.21b)$$

$$\int_{0}^{l\max}4f\frac{dx}{D}=\int_{M}^{1}\frac{1-M^{2}}{rM^{4}\left[1+r-1/2\,M^{2}\right]}dM^{2}$$

Where the limit of integration are taken at (i), the section where the mach no is M and where x is arbitrarily set to zero, and (ii) the section where mach no is unity and x is the maximum possible length of duct (lmax).

Carrying out the integration

$$4\vec{f}\frac{l\max}{D}=\frac{1-M^{2}}{rM^{2}}+\frac{r-1}{2r}\ln\left[\frac{(r+1)M^{2}}{2\left(1+\left(r-1/2\right)\right)M^{2}}\right]\text{............}(4.22)$$

$$\text{Where }\vec{f}=\frac{1}{l\max}\int_{0}^{l\max}fdx\text{ the mean friction coefficient}$$

Length of duct (required to pass from a given mach number, M_1 to a final mach number M_2 is found to be

$$\frac{(4fl)}{D}=\left(\frac{4fl\max}{D}\right)_{M_1}-\left(\frac{4fl\max}{D}\right)_{M_2}\text{ ....}(4.23)$$

To find P VS M

Substitute for dV/V using (4.17) and (4.20) and also for 4f dx/D from (4.21a) in (4.14) to get

$$\frac{dP}{P}+\frac{1-M^2}{1+\frac{r-1}{2}M^2}\frac{dM}{M}-\frac{\frac{1}{2}rM^2(r-1)M^2}{1+\frac{r-1}{2}M^2}\frac{dM}{M}$$

Collecting terms

$$\frac{dP}{P}=-\frac{dM}{M}\left[\frac{1+(r-1)M^2}{1+\frac{r-1}{2}M^2}\right]\ldots\ldots\ldots\ldots\ldots\ldots(4.24)$$

Integrating between limits P and P*, where superscript *, denote the property at mach 1

$$\int_P^{P*}\frac{dP}{P}=-\int_M^1\frac{dM}{M}\left[\frac{1+(r-1)M^2}{1+\frac{r-1}{2}M^2}\right]$$

$$\frac{P}{P*}=\frac{1}{M}\left[\frac{r+1}{2\left(1+\frac{r-1}{2}\right)}\left(M^2\right)\right]^{1/2}\ldots\ldots\ldots\ldots\ldots\ldots(4.25)$$

Using similar methods, the formulae which follow can be obtained

$$\frac{dV}{V}=\frac{rM^2}{2\left(1-M^2\right)}4f\frac{dx}{D}\ldots\ldots\ldots\ldots\ldots(4.26)$$

$$\frac{dT}{T}=\frac{1}{2}\frac{dC}{C}=-\frac{r(r-1)M^4}{2\left(r-M^2\right)}4f\frac{dx}{D}\ldots\ldots.(4.27)$$

$$\frac{d\rho}{\rho}=-\frac{rM^2}{2\left(r-M^2\right)}4f\frac{dx}{D}\ldots\ldots\ldots\ldots(4.28)$$

$$\frac{dP_t}{P}=-\frac{rM^2}{2}4f\frac{dx}{D}\ldots\ldots\ldots\ldots(4.29)$$

After substituting for 4f dx/D in equation (4.26) – (4.29) they can be integrated to get

$$\frac{V}{V^*} = M\left[\frac{r+1}{2\left(1+\frac{r-1}{2}\right)}M^2\right]^{1/2} \quad \text{............}(4.30)$$

$$\frac{T}{T^*} = \frac{C}{C^*} = \left[\frac{r+1}{2\left(1+\frac{r-1}{2}\right)}M^2\right] \text{............}(4.31)$$

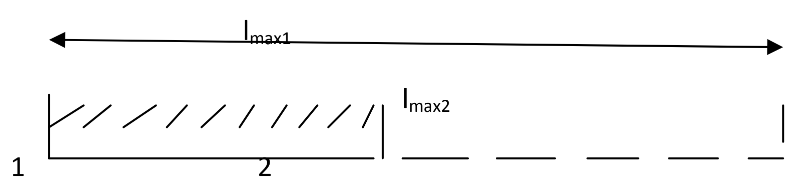
$$\frac{\rho}{\rho^*} = \frac{V^*}{V} = \frac{1}{M}\left[\frac{2\left(1+\frac{r-1}{2}M^2\right)}{r+1}\right]^{1/2} \quad \text{............}(4.32)$$

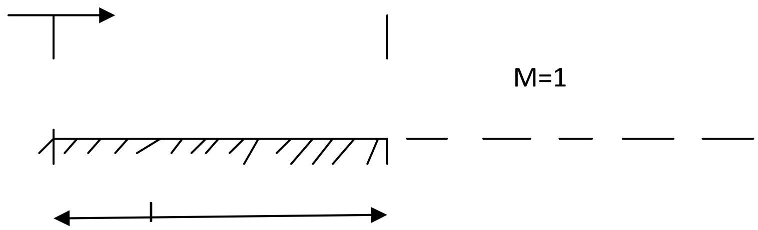
$$\frac{P_t}{P_t^*} = \frac{1}{M}\left[\frac{2\left(1+\frac{r-1}{2}M^2\right)}{r+1}\right]^{1/2} \quad \text{..............}(4.33)$$

Equation (4.22), (4.25), (4.30) – (4.32) are tabulated in table B4.

Example

Flow enter a constant area duct with a mach number of 0.6, static pressure 10psia and static temperature 500^{0}R, assume a duct length of 15”, diameter 1” and a friction coefficient of 0.005, determine the mash number, static pressure and temperature at the duct outlet?

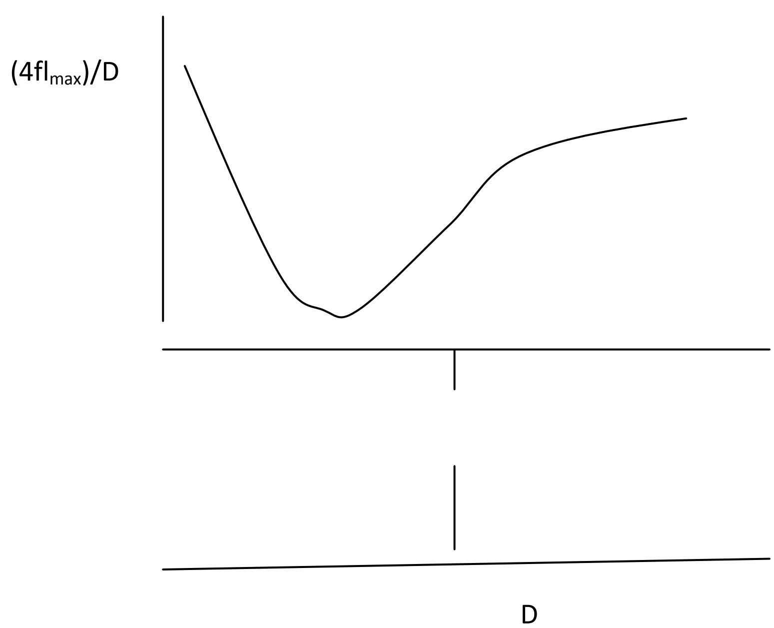




From B4, at M_1=0.6, $\left[\frac{(4fl_{\max})}{D}\right]_1 = 0.491$

Actual $\frac{4fl}{D} = 0.30$

$$\left[\frac{(4fl_{\max})}{D}\right]_2 = \left[\frac{(4fl_{\max})}{D}\right]_1 - \frac{4fl}{D} = 0.491 - 0.300 = 0.191$$



$M_2 = 0.71$

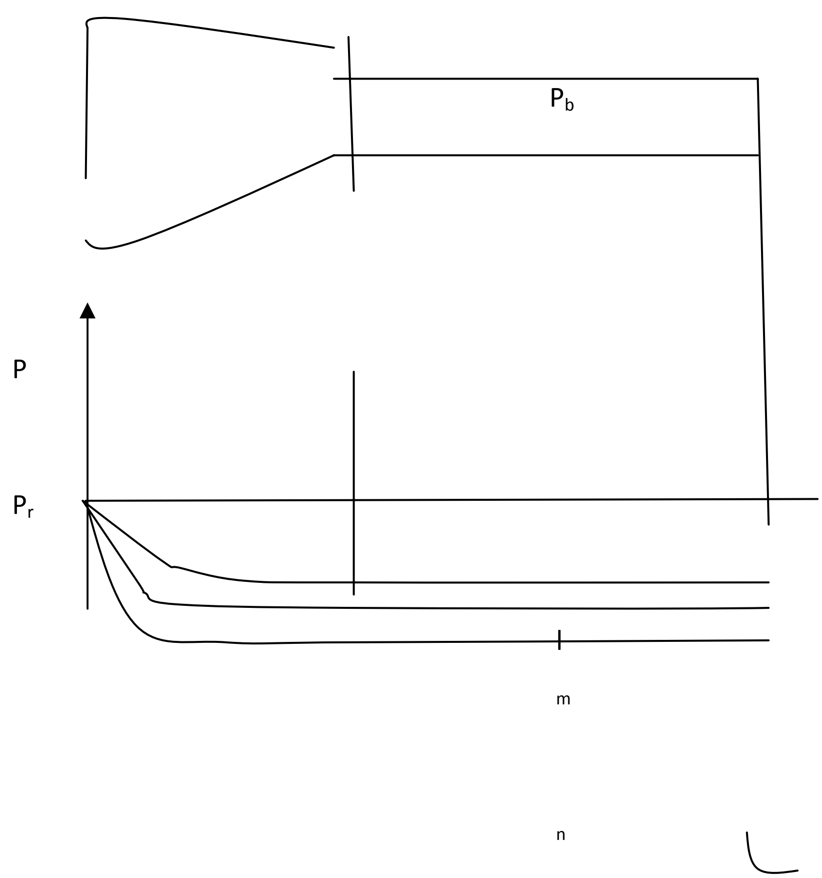
$$\frac{P_2}{P_1} = \left(\frac{P_2/P^*}{P_1/P^*} \right) = \frac{1.471}{1.763} = 0.834$$

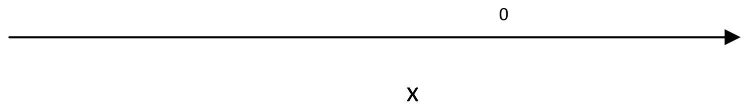
$$P_2 = 8.34\, psia$$

$$\frac{T_2}{T_1} = \left(\frac{T_2/T^*}{T_1/T^*} \right) = \frac{1.0919}{1.119} = 0.97$$

Flow through a nozzle and constant area duct in series

(a) Converging nozzle



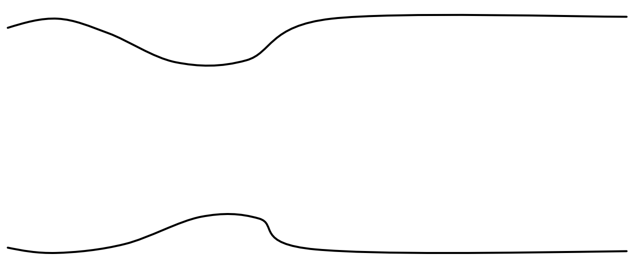


P_r is maintained constant.

Assume isentropic flow in the nozzle, and fanno flow in the duet, varying P_b.

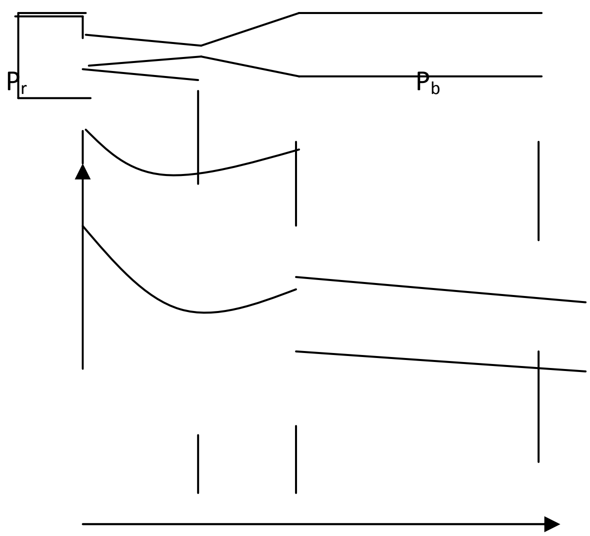
As P_b is lower below P_r, curves such as L and M are obtained, with pressure decreasing in both nozzle and duct. When P_b is decrease to that of curve n, mach 1 occur at duct exit. Further decrease in P_b cannot be sense by the reservoir for all P_b below that of curve n and the mass flow rate remain the same as that of curve n. The system is choked by the duct and not by the nozzle and the maximum mass flow rate in this system is less than that   for the same reservoir pressure with a converging nozzle only.

(b) **Converging – diverging nozzle**

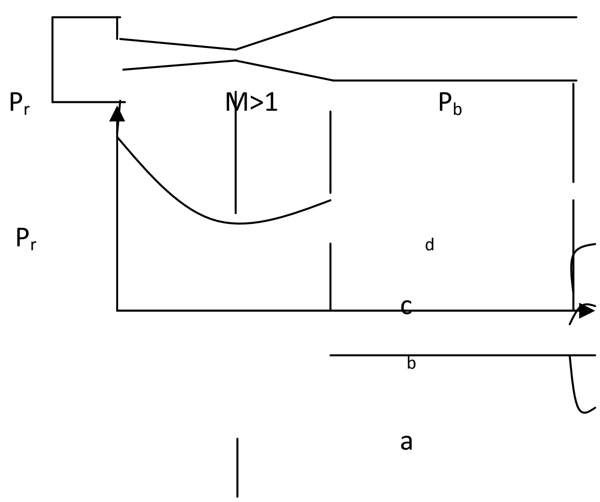


Depending on the duct length the minimum pressure point or point of maximum mach no can occur at the nottle throat or duct exit.

Case 1: If the duct is long enough, the system reaches mach 1 first at the duct exit, the nozzle is not choked. Super sonic flow is impossible.



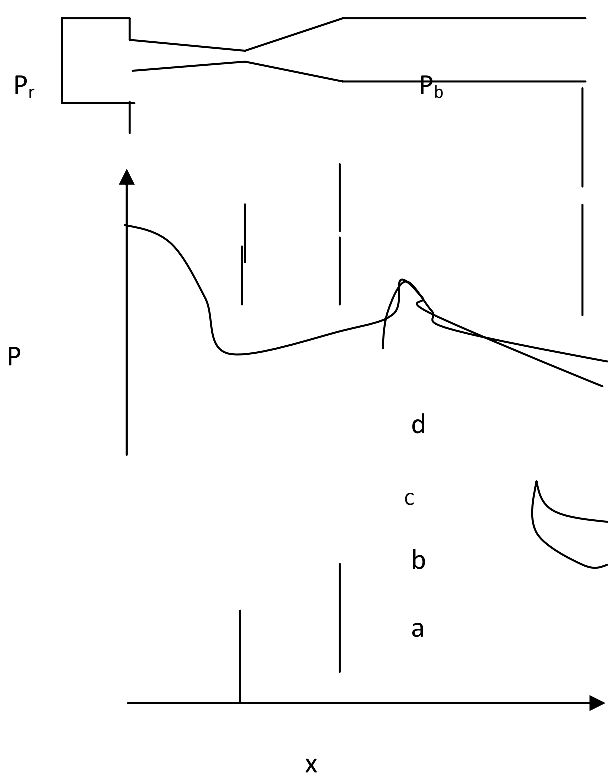
Case 2: If duct is not long enough to give mach 1 at exit, supersonic flow can occur in the nozzle exit. Once P_b is low enough to produce mach 1 at the nozzle throat, the system is chocked by the nozzle and no further increase in mass flow rate is possible. If the length of the duct is less than that which will decelerate the flow to mach 1 at the duct exit (lmax) , supersonic flow occurs at the duct exit. Consider changes in the duct flow as successively higher, P_b values are chosen from P_b = 0 absolute.



x

l < lmax corresponding to M > 1 at duct inlet for a back pressure as low as that at curve a, an expansion wave is formed at the duct exit. When P_b is raised to be equal to that of curve b, the exit plane pressure is equal to the back pressure. A further increase in P_b yields obligue shock wave at the duct exit until eventually, for a P_b equal to that of curve d a normal shock stand at the duct exit. Increases in P_b over that cause the shock to move into duct.

For l > lmax corresponding to M > 1 at duct inlet



A normal shock wave appears in the duct. For a back pressure that is very low (close to 0pascal) the back pressure is less than the exit plane pressure (P_E) so that expansion waves must occur at the duct exit with the exit plane mach number equal to 1. This is the case for curves a and b. For curve c, P_e equal $P_{b.}$ As P_b is raised above that of curve c,the normal shock moves towards the duct inlet, and the exit mach number is subsonic with $P_E = P_b$. For a P_b that is high enough, the normal shock moves into the nozzle eliminating supersonic flow in the duct.